USING GROUP-WORTHY TASKS TO IMPROVE STUDENT FLEXIBILITY WITH MATHEMATICAL REPRESENTATIONS

by

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Abstract

Historically high school math students have struggled to move flexibly between algebraic, tabular, descriptive, and graphical representations. This deficit results in thin conceptual understanding and serves as a roadblock to students' pursuit of future coursework in mathematics. Current research provides suggestions for improving student understanding with representations but little research has measured student growth in this area following a pedagogical change. This study looks at a group of average-achieving high school students in a precalculus class and how the implementation of representation oriented group-worthy tasks profoundly impacted student understanding of algebraic, tabular, descriptive and graphical representations of functions. Using a matched-pairs t-test students in the experimental group made statistically significant gains while students in the comparative group, who did not use group-worthy tasks, saw no statistically significant change.

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Chapter I: Introduction and Literature Review

Rationale

Through my experience in a variety of different school settings, I have worked with mathematics teachers that refuse, half-heartedly implement, or embrace research-based practices. Generally speaking, the justification for resistance is based on the supposition that trends in research are just that – trends. However, research on the effectiveness of cooperative learning (or groupwork) has been confirmed repeatedly for decades (Slavin, 1991). Many classroom teachers, including myself, find cooperative learning is a powerful tool for improving thinking skills, developing conceptual understanding, addressing heterogeneous ability groups, and mediating achievement differences based on status (Boaler & Staples, 2008; Cohen & Lotan, 1995; Slavin, 1991).

I have worked vehemently in my own classroom to integrate groupwork using Cohen's framework – training students to work in groups, setting norms, assigning roles, and creating group tasks (1994). I have witnessed student growth in helping and affirming others, time on task, ability to justify, and, most importantly, improved conceptual understanding. Thus, the effectiveness of cooperative learning has been confirmed for me personally through my classroom experience.

While I am overjoyed by the improved depth of understanding in my students, one area of weakness persists: linking representations. Many of my students move with ease from a function to a graph but their ability is lacking when moving between other representations such as the graph-function direction, writing functions for tables, using graphs to answer questions about functions, or recognizing the power of different representations in different contexts. What astounds me is that as I have continued to work with my former students (by tutoring them in their college courses) I see this lack of representational fluency impeding their ability to progress in mathematics at the college level – especially for students taking calculus. Even with the simplest use of function notation, these former students, now freshmen and sophomores in college, are baffled. And, interestingly, I see this weakness with a range of ability levels. To clarify the struggles students experience, I have provided a basic example in Figure 1.1.

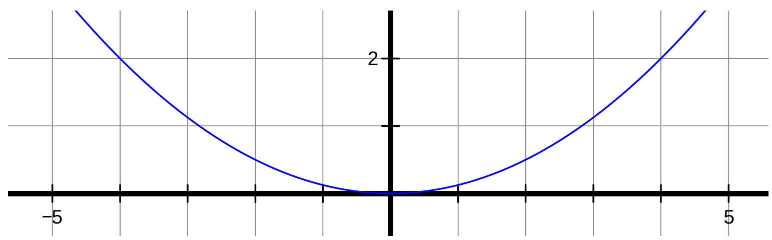


Figure 1.1. Example linking representations question. The graph of $f(x)$ is shown. If $f(x) = 2$, what is a possible value for x?

Note: Leaf, B. (2010). *Top 50 Sklls for a Top Score SAT Math.* New York: McGraw Hill.

Most students struggle with knowing where to begin with this type of question. When teaching an SAT class to a wide range of achievers, including those above 75th percentile, the question in Figure 1.1 was an automatic "omit" for students. This means they were so unsure about what the question was asking that students chose to not even attempt the problem. While some may think that students struggle with this type of question solely because of the function notation, I would argue that student struggles can be attributed to the disconnect between representations. Just prior to the problem in Figure 1.1 students were evaluating functions with ease; thus, my observation is that few students see any connection between a function, its graph, and its solutions. More importantly, students view these entities as individual objects within the object-specific domain rather than looking at these entities as different representations of the same thing (Moschkovich, Schoenfeld, & Arcavi, 1993).

Because I have witnessed the power of groupwork in my classroom, and because I feel linking representations with functions continues to be an area of weakness, I have decided to explore how I might successfully link the two.

Through my research I will explore the following question: How can group-worthy tasks impact student flexibility with mathematical representations when studying functions?

I decided to conduct this research while students studied two units: one on linear functions and another on general function transformations. In order for students to have access to the varied representations, I felt it was important to start with an accessible area of focus (linear functions). Even lower achieving students in my class had some familiarity with writing and graphing linear functions. Also, the topic of functions is an integral part of mathematical understanding and many students find it challenging. Coupling the improvement of representational fluency and function understanding could serve as an empowering building block for students.

In considering the research question, it is important to define what is meant by groupwork, group-worthy tasks, and mathematical representations. Thus, the sections that follow describe how I will define these key terms.

Cooperative Learning or Groupwork

Cooperative learning is much more than moving desks and asking students to work collaboratively. The achievement gains purported in cooperative learning research describe many additional classroom and pedagogical factors such as group assignment, training students to work cooperatively, adapting or creating tasks for groupwork, and monitoring group interaction (Cohen, 1994).

Students must be trained to work productively in groups. Group training includes group-building activities, setting norms for behavior, assigning roles, and teaching students how to communicate understanding through organized and productive discussion (Cohen, 1994; Gillies, 2003; Prichard, Stratford, & Bizo, 2006). Students also must learn listening skills: what constitutes listening behavior, how to ask and answer questions, and how to synthesize the ideas of the group (Cohen, 1994).

Furthermore, it is imperative to attend to group size and composition. When constructing groups, four to five students per group is optimal as well as heterogeneous composition by ability and status (Gillies, 2003). Too small of a group can result in less input for approaching a complex problem and too large of a group usually results in students being left out. When a task is designed for multiple abilities, mixing students by ability results in greater access to increased achievement for low achieving students (Cohen, 1994; Slavin 1991). Cohen (1994) and Boaler (2008) assert mixed ability student groups result in higher frequency of discussion which, in turn, results in greater depth of understanding for all students. Some researchers have argued groups should be either equally male and female or all male and all female because, in majority-male or majority-female groups of similar ability, the boys obtained higher learning outcomes than the girls (Gillies, 2003). Further, Lou et al. (1996) found that low achieving students benefited most from heterogeneous groups, high achieving students learned equally well in either hetero- or homogeneous groups, and middle level students benefited significantly more in homogenous groups. Hence, there is much to consider when creating groups in the classroom.

The teacher's role in groupwork is much different than in a classroom predominantly led by direct instruction – the mathematical authority is now more equally shared between the teacher and the student. When teachers create group tasks as a means to learn they "delegate intellectual authority to their students and make their students' life experiences, opinions, and points of view legitimate components of the content to be learned" (Lotan, 2003, p. 72). The students no longer seek the immediate affirmation of the teacher. Instead, the groups of students work together to discuss their thinking, defend their ideas, and arrive at different solutions (Lotan, Cohen, & Holthuis, 1994). Ideally the teacher provides very little intervention as the students work together to arrive at solutions. Too much direct supervision by the teacher can result in less student learning. When a teacher interrupts the process of the task through instructing, disciplining, or telling students how to get through a task, the productivity of the student group declines greatly; in turn, as the productivity of the student group declines, so too does the learning (Lotan, Cohen, & Holthuis, 1994). What's more, while a task can be written with every intention of eliciting complex thinking with high cognitive demand, the demand can decrease based on specific classroom factors. Stein & Henningsen (1997) found the greatest factors to maintaining strong cognitive demand within a task were (1) building on prior knowledge, (2) scaffolding less, (3) allotting appropriate time, (4) modeling high level performance, and (5) sustaining pressure for explanation and meaning. Through Stein & Henningsen's research it was determined that while a task might have outstanding potential with high cognitive demand, students can regress to procedural thinking without connection to meaning, unsystematic exploration, or even no mathematical activity when the teacher does not focus on the aforementioned five factors. Once a group task declines, the potential for deeper learning and understanding through discussion and justification is lost.

While some may think the teacher takes a lesser role when moving from direct instruction to collaborative learning, the research suggests quite the opposite. The teacher selects the task, has trained students to work in groups, assigns roles, keeps students accountable to adhering to the roles during the tasks, oversees the task to see that the intended learning is carried out, and works with the whole class to discuss questions and summarize the learning of the task. I would argue the role of the teacher during groupwork is more demanding than the role of the teacher during direct instruction.

Group-Worthy Tasks

Once the stage for groupwork is set, the greatest burden for the teacher becomes the development of a task. When students work in groups to hash out procedural problems the power of the group learning is lost – the task can be completed individually, there is only one correct answer, and not all students have access to the task (Cohen, 1994). Rather,

> A group task is a task that requires resources (information, knowledge, heuristic problem-solving strategies, materials and skills) that no single individual possesses so that no single individual is likely to solve the problem or accomplish

> the task objectives without at least some input from others. (Cohen B. & Arechavala-Vargas, 1987 as cited in Lotan, Cohen, & Holthuis, 1994, p. 4)

The development of a task is fundamental for building interdependence, justification, and conceptual understanding. The goal of a group task should not be completion; rather, students should be working together to develop new learning (Slavin, 1991). Figure 1.2 gives five design features used to deem a task group-worthy. These types of problems allow students to truly grapple with concepts through exploration and discussion. Cohen

- They are open-ended and require complex problem solving.
- They provide students with multiple entry points to the task and multiple opportunities to show intellectual competence.
- They deal with discipline-based, intellectually important content.
- They require positive interdependence as well as individual accountability.
- They include clear criteria for the evaluation of the group's product.

Figure 1.2. Group-Worthy task design features. Lotan, R. A. (2003, March). Group-Worthy Tasks. *Educational Leadership* , p. 72.

(1994) describes these tasks as *equal-exchange* (p. 64) as students have to share ideas and thinking because completing the task individually would be far more difficult. Further, in writing the tasks teachers must resist the temptation to spell out all steps. If a task is written with too many explicit instructions from the teacher, the critical thinking necessary to address the problem is removed (p. 71). Inherent in Lotan's description of group-worthy tasks is the attention to cognitive demand.

Cognitive demand is a means to describe the complexity of a problem or task. We use four descriptors for levels of cognitive demand with specific features at each level – these are included in Figure 1.3. Essentially, these levels describe how students might think or reason about the mathematics being presented. When a task is well developed –

attending to the features of a group-worthy task and high-level cognitive demand – it supports the content being studied and helps students make sense of major ideas (Cohen, 1994).

Lower-level demands (memorization)

- Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory.
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.

Lower-level demands (procedures without connections to meaning)

- Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
- Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used.
- Are focused on producing correct answers instead of on developing mathematical understanding.
- Require no explanations or explanations that focus solely on describing the procedure that was used.

Higher-level demands (procedures with connections to meaning)

- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

Higher-level demands (doing mathematics)

- Require complex and non-algorithmic thinking – a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.
- Require students to explore and understand the nature of mathematical concepts, processes, or relationships.
- Demand self-monitoring or self-regulation of one's own cognitive processes.
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.

Figure 1.3. Levels of cognitive demand. Arbaugh, F., & Brown, C. A. (2005). Analyzing mathematical tasks: a catalyst for change? *Journal of Mathematics Teacher Education , 8*, p. 530.

Multiple Representations

Even (1998) describes the conceptual benefits of strong representational understanding:

> The ability to identify and represent the same thing in different representations, and flexibility in moving from one representation to another, allows one to see rich relationships, develop a better understanding, broaden and deepen one's understanding, and strengthen one's ability to solve problems (p. 105).

Representations can be described as the means by which students are able to record and analyze patterns. If a student cannot flexibly convert between different representations, they will struggle to analyze function relationships, find similarity in structure, identify real-world situations that can be modeled by the same class of function, convey meaning in different ways, or develop deeper understanding (NCTM, 2004). Furthermore, students must be able to identify the important features of different representations. If a student can successfully use the most appropriate representation, they will gain competence and become more efficient problem solvers (Moschkovich, Schoenfeld, & Arcavi, 1993).

When a student is mathematically proficient, they possess strategic competence and adaptive reasoning, both of which include features of representation. Strategic competence means students are able to analyze a problem, build a representation, and then identify commonalities of that representation with other mathematical structures. "Expert problem solvers focus more on the structural relationships within problems, relationships that provide clues for how problems might be solved" (Ball, 2003, p. 125). With adaptive reasoning representation-building helps students develop sophisticated reasoning abilities (Ball, 2003).

In order for students to successfully navigate the mathematical terrain, they must become expert problem solvers through reasoning and sense making. The process strands developed by NCTM – Problem Solving, Reasoning and Proof, Connections, Communication, and Representation – are "all manifestations of the act of making sense of mathematics and of reasoning" (NCTM, 2009, p. 5). As such, representation is a key component to developing student understanding. More specifically, NCTM describes key elements of reasoning and sense making specific to functions and representations as shown in Figure 1.4. NCTM then breaks down the knowledge of functions into five

1. Representing functions in various ways – including tabular, graphic, symbolic (explicit and recursive), visual, and verbal
2. Making decisions about which representations are most helpful in problem-solving circumstances
3. Moving flexibly among those representations

Figure 1.4. Key elements of reasoning and sense making specific to functions and representations. NCTM. (2009). Reasoning with Functions. In *Focus in High School Mathematics-Reasoning and Sense Making* (pp. 41-53). Reston: National Council of Teachers of Mathematics.

essential understandings. The fifth essential understanding pertains to multiple representations (NCTM, 2009). The descriptors given in Figure 1.5 are major areas of focus under Big Idea 5. These essential understandings further describe what it means for

Essential Understanding 5a. Functions can be represented in various ways, including through algebraic means (e.g., equations), graphs, word descriptions and tables.

Essential Understanding 5b. Changing the way that a function is represented (e.g., algebraically, with a graph, in words, or with a table) does not change the function, although different representations highlight different characteristics, and some may show only part of the function.

Essential Understanding 5c. Some representations of a function may be more useful than others, depending on the context.

Essential Understanding 5d. Links between algebraic and graphical representations of functions are especially important in studying relationships and change.

Figure 1.5. Big Idea 5. NCTM. (2010). *Developing Essential Understanding of Functions Grades 9-12*. Reston: National Council of Teachers of Mathematics.

a student to have flexibility with representations when studying functions.

Not only is the use of mathematical representations evidence of students' depth of understanding, but the process of developing representations in itself promotes learning. For one, each representation carries with it features that illuminate different aspects of a problem; consequently, the information gained from combining representations is greater than any single representation. Second, different representations help constrain each other by limiting the contributing pieces to the problem and narrowing the focus of the analysis for the students. Finally, when students have to relate representations to each other they are engaging in activities that promote deeper understanding (Friedlander & Tabach, 2001).

Representational understanding begets overall concept understanding. "The fundamental goals of mathematics education include representational goals: the development of efficient internal systems of representation in students that correspond coherently to, and interact well with, the external conventionally established systems of mathematics" (Goldin & Shteingold, 2001, p. 3). It is important to be cognizant of students' internal and external representations. The external representations are the signs, characterizations, and objects that stand for something other than itself (such as a graph for a data table) while the internal representations are verbal/syntactic, imagistic, formal notation, and affect. When a student has fully developed their understanding of a concept, then they have internal *and* external representations (Goldin & Shteingold, 2001). Moreover, when a student has developed appropriate internal representations, as well as the relationships between each representation, then they have learned and can apply a mathematical concept.

The external and internal representations described by Goldin & Shteingold (2001) support previous descriptions of representational understanding through four modes: cognitive and perceptual, explanatory with modeling, representations within mathematics, and external symbolic representation (Kaput, 1987). However, Kaput claims that the

> Fundamental premise is that the root phenomena of mathematics learning and application are concerned with representation and symbolization because these are at the heart of the content of mathematics and are simultaneously at the heart of the cognitions associated with mathematical activity (p. 22).

What Kaput has clearly described is a sort of gatekeeper for mathematical understanding. Giving students access to complex problems and higher order thinking will fall short if students do not have the representational and symbolic fluency.

As students begin to problem solve using representations, it is important not only to identify the students' final product but the understanding exhibited through the representation. For instance – did the student create a graph through a global or point-wise approach? In a global approach students can identify patterns and key behaviors of functions. If a students' understanding is point-wise, they see graphs as a collection of discrete points. Students will fixate on one way to approach problems – they are dominated by either the global or the point-wise approach. However, flexibility between these two ways of thinking is necessary for students to successfully determine the best way to investigate a problem situation. Interestingly, students with a global approach are more successful with problems in different contexts, but struggle to identify the point relationships within the graph of a function they are examining (Even, 1998).

Another means used to describe the differences in student understanding of graphs and functions is known as the Cartesian connection which represents students' ability to move in an equation-to-graph and graph-to-equation direction (Knuth, 2000). Unfortunately, research studies have found the ease with which students can move in both directions is quite limited (Even, 1998; Gagatsis & Shiakalli, 2004; Goldin & Shteingold, 2001; Hitt, 1998; Knuth, 2000; Meij & Jong, 2003; Rider, 2007). Much research has indicated that when students are presented with a problem that would be easier to solve using a graphical approach, they will persist in using an algebraic approach (Aspinwall, 2007; Even, 1998; Knuth, 2000; Rider, 2007). Further, students will actually perceive graphical representations as unnecessary and often unconnected to the corresponding algebraic representation. This also means students are unable to justify a solution to a problem using an alternative representation (Knuth, 2000). While student understanding of representations is a necessary component to deeper concept understanding, the research demonstrates students continue to struggle in this area.

Group-Worthy Tasks and Representations

Instruction in mathematics must include representations with written descriptions, algebraic forms, tables, and graphs. If the instructional approach with representations is not varied, then students' ability to move flexibly between representations will also suffer (Aspinwall, 2007; Even, 1998; Knuth, 2000(a); Knuth, 2000(b); Pyke, 2003). Several research studies have focused on improved student learning through the use of collaborative learning (Blatchford, 2003; Boaler & Staples, 2008; Cohen & Lotan, 1995; Prichard, Stratford, & Bizo, 2006; Slavin, 1991), developing group-worthy tasks (Gillies, 2003; Lotan, 2003), and presenting student groups with cognitively demanding tasks

(Lotan, Cohen, & Holthuis, 1994; Stein, Grover, & Henningsen, 1996; Stein & Henningsen, 1997;). However, none of the research explores how collaborative learning and group-worthy tasks could be designed to target student improvement in one focused area.

Because the research about collaborative work and group-worthy tasks has been more broad, I thought it would be worthwhile to develop tasks targeting the unit content while, at the same time, embedding an essential understanding as described by NCTM (2010). This means each group task focused on the unit learning targets while also integrating multiple representations. For instance, one of the unit learning targets was "identifies and writes functions with vertical and/or horizontal shifts, reflections, and vertical/horizontal stretches and compressions." While the group task developed understanding around this learning target, the task also presented different representations, required students to connect representations, and finally asked students to present solutions using different representations. Thus, each group-worthy task had two goals: 1) develop student understanding around the unit content and 2) build student knowledge of, and flexibility with, multiple representations. Because I had experienced students' persitent weakness with representations, and because I experienced the learning power of group-worthy tasks, I felt it would be worthwhile to marry the two ideas. As a result, I decided on the following research question: How can group-worthy tasks impact student flexibility with mathematical representations when studying functions?

Chapter II: Methods

Participants & Setting

The setting is a private independent Catholic school in a suburban community. The surrounding city population is 40,670 with a median household income of $101,592 (2009 census data). Because the school is private, the student body represents 140 area public schools from 41 different zip codes. While the majority of students are Caucasian from middle and upper SES families, 30% of students receive tuition assistance and 13% of the study body is African-American, Asian, or Mexican-American.

The math classes at this school are tracked: there is a regular level and an honors level course at each step in the progression. Students are placed by readiness (Algebra, Geometry, etc.) into either track based on standardized test scores, previous course and course grade, teacher recommendation, and a school written math placement exam. The participants in this study are students in the lower track precalculus course. Higher track students are enrolled in an honors precalculus course taught by a different teacher. The two different courses use the same text, but the lower track course covers fewer chapters and fewer concepts. Essentially, the course is intended to provide students with a stronger foundation of mathematics before going on to college. This is especially important because, as the school website claims, 95-98% of students in this school attend post-secondary education institutions.

While the students in this class are enrolled in precalculus, they are not high mathematics achievers. Achievement on standardized test scores for students in the precalculus course is much lower than students in the Honors course. The box plot shown in Figure 2.1 gives a comparison between the current seniors in the honors course versus the current seniors in the lower track precalculus course for the 2009 PSAT. The PSAT/NMSQT data shown reflects math scores alone. In 2009, the national average math score for juniors in high school was 48 (http://www.collegeboard.com/student/testing/psat/scores.html). The median score for the precalculus group was 50 (55th percentile) while the median score for the honors group was 58 (78th percentile). Thus, the precalculus students are average achievers. Also, as seen in Figure 2.1, the precalculus group includes students at very low PSAT achievement levels. All students at this school are required to take the PSAT. Thus, the data is representative of the entire population.

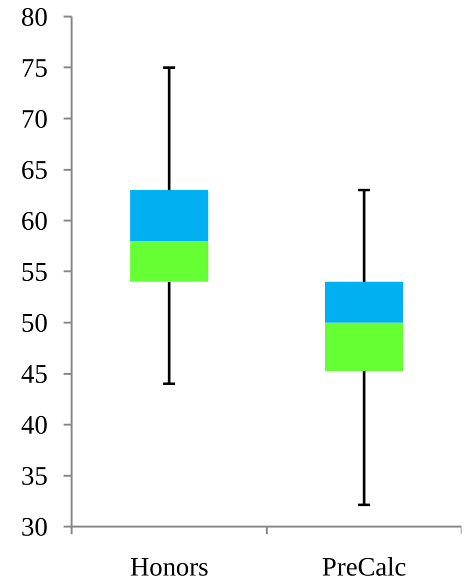


Figure 2.1: Box plot comparison of PSAT performance by class.

The achievement of students in each class measured on a national test is important to include for a few reasons. For one, some may argue the success of the participant group is not transferrable to average or low achieving groups in a more typical school environment. I would argue that while these participants attend a private school, and come from higher SES families, their mathematics achievement is truly average. Thus, the ability to replicate the success of this research study outside a private school setting is indeed plausible. Additionally, the PSAT data helps to substantiate the difference in performance between the honors and precalculus classes on the pretest.

The past learning experience of the student participants could be described as traditional math instruction. This means students have encountered mostly direct instruction, little to no group work, tasks predominantly at the memorization or procedures without connections level of cognitive demand, and summative assessments frequently requiring only procedural understanding. Thus, learning in groups with far less direct instruction in a mathematics course is a novel experience for almost all participant students.

I selected this group of students for my research study because many are struggling learners. Many have not had positive learning experiences in mathematics and possess little self-efficacy. In a different school setting (not college preparatory), students with similar math achievement might not take a 4^{th} year of mathematics in high school. Because almost all students in this class will attend a 4 year college or university, it is imperative that they leave my class ready to both successfully place into college mathematics and complete a college level math course. Flexibility with mathematical representations improves depth of conceptual understanding and lays the foundation for advanced content at the college level. Thus, I felt this population of students could benefit most with a focus on improving their understanding of representation through my research study. Furthermore, this particular group of students will help answer my research question because they have little experience working in groups in a math classroom and have very little flexibility with representations.

Data Collection

Four different sources of data were collected and analyzed: group tasks, individual follow-up tasks, pre and posttest performance, and student interviews. It is necessary to analyze the research question through a variety of data sources. While pre and posttest data might result in numerical results, it does little to shed light on what students are thinking about while wrestling with representation problems. Further data sources to address the research question support triangulation which, in turn, makes research findings more credible (Mertens, 2010).

Within each unit of instruction I provide students with a list of learning targets. These are objectives which are tied to every lesson, task, and assessment. Thus, prior to starting the research process I thought about the representation learning targets I would assess throughout my data collection. I wrote the learning targets with the representations pretest. I felt it was important to clearly outline the objectives my students would be working toward so I could clearly compare performance as the targets threaded throughout the research study. The list of learning targets is given in Figure 2.2.

1	Evaluates functions using a table
2	Identifies linear functions from a table
3	Uses tables to approximate solutions
4	Uses graphs to approximate solutions
5	Identifies multiple solution strategies
6	Writes linear equations from tables
7	Evaluates functions using a graph
8	Finds x values for a given function value on a graph
9	Evaluates composite functions using a graph
10	Writes an equation from given function values
11	Interprets a situation described by a graph
12	Interprets a description to create a graph
13	Translates a written description into a function
14	Connects graphs to solutions of functions

Figure 2.2. Research learning targets pertinent to student work with representations.

Each group-worthy task used and/or developed for the unit was evaluated using Lotan's five criteria and the measures of cognitive demand (Figures 1.2 and 1.3). It is important to apply these evaluative measures because the research findings are based on the premise that group-worthy tasks will impact student flexibility with representations. Additionally, I want to make my rationale in selecting each task transparent for transferability (Mertens, 2010).

On the fourth day of class students took a pretest assessing their ability to answer questions pertaining to mathematical representations. Each test question was aligned to a different learning target from Figure 2.2. The pretest questions were carefully selected and/or written to determine student connections between tables, graphs, function notation, solutions to equations, and verbal descriptions. The test questions were adapted from a variety of different sources. A copy of the pretest with references for sources is also included in Appendix A. The content on the pretest was material students had studied in Algebra and Advanced Algebra courses. All of the questions pertained to simple functions (mostly linear) which students had practiced extensively in previous years. However, the questions were written to elicit understanding about representations. The intent in using these types of questions was that the material was familiar but asked in perhaps an unfamiliar context.

Each group-worthy task targeted specific research learning targets. To measure student learning following each group-worthy task, students completed an individual assessment either immediately following the group work or first thing the next class day. The individual tasks ranged from brief exit questions to formative quizzes or chapter exams. Student progress throughout the unit was then tracked by learning target. I tallied correct and incorrect responses for the individual tasks and tracked the percentage of students who answered questions for each learning target correctly over the course of the data collection process.

Following completion of the two functions units students completed a posttest. The pre and posttest questions were not the same. To address possible problems of comparability with using different questions sets, each question is aligned to the same set of learning targets and the same grading criteria was applied to both the pre and posttest. A copy of the posttest is included in Appendix B. Each student pre and posttest score was paired and a difference in total score was calculated. Following, a matched pairs t-test was used to test significance.

Once the posttest was completed the final data collection method was student interviews. Following the completion of the posttest I congratulated students on their gains and then posted a list of student numbers representing those students whom I felt were appropriate to interview. I asked for volunteers. I provided a list of specific student numbers because I wanted to speak with students who were within 2 standard deviations of the mean posttest score – this would encompass 95% of the students in my class. The top students, with a z score greater than 2, are likely learning regardless of use of group-worthy tasks and group work. The lower students, with a z score less than -2, are those that I intervene with on a one-on-one basis and are outside the "normal" scope of my instructional approach with group work in my classroom.

I ended up obtaining consent forms from five different students: Ashley, David, Paul, Lucas, and Kim. The pre and posttest data for each student is given in Figure 2.3. Ashley was the one interview participant who did not complete the pretest; however, she wanted badly to participate in the interview. I decided not to exclude her because she felt

	Ashley	David	Paul	Lucas	Kim
Pretest	---	32%	47%	50%	53%
Pretest z-score	---	- 0.5	0.57	0.78	1.0
Posttest	28%	86%	72%	69%	78%
Posttest z-score	-1.78	1.4	0.67	0.69	1.0

Figure 2.3: Interview participant data.

it was important to share her experiences. While all of the students made gains in their overall scores, there are differences of significance when looking at the pretest z-scores and the posttest z-scores. So I felt this was a reasonable cross section of students in my class for the interview.

The interviews followed a semi-structured format with some guiding questions and additional questions that followed from student responses (Mertens, 2010). The interviews took place either during lunch or at the end of the school day. Each interview was recorded and then transcribed for thematic analysis. Originally my intent was to address my research question directly by linking group work, group-worthy tasks, and flexibility with representations. In conducting the interviews, though, I found this was far too difficult a connection for students to make without me hand-feeding them the idea. So as I progressed through the interviews the questions pertained more to students' prior experiences in math classes and their perceptions of the effectiveness of using group-worthy tasks with the intent of improved learning. The interview questions are included in Appendix C.

Through each data collection method secondary research questions arose. Each secondary question supports the overarching research question regarding group-worthy tasks and flexibility with representations. Table 2.1 gives each method with its relationship to the different secondary research questions.

Table 2.1

Secondary Research Questions and Data Collection Methods

Research Question: How can group-worthy tasks impact student flexibility with mathematical representations when studying functions?				
	DATA COLLECTION METHODS			
Secondary Research Questions	Pre and Posttest	Individual Tasks	Classroom Observations	Student Interviews
How do group-worthy tasks enhance student learning?		☑	☑	☑
In what ways do students identify the strengths and weaknesses of different representations?	☑	☑	☑	☑
What do students understand about linking and moving between representations?	☑	☑	☑	☑

Note. A ☑ in a cell signifies that results from that specific data collection method help to address the corresponding secondary research question.

Many steps are involved in the data collection process and I was careful to plan out these steps according to a timeline. The function units spanned from August to November. The posttest was given in November and the student interviews were conducted in early December. A timeline of the study is outlined in Table 2.2.

Table 2.2

Research Timeline.

August	**September**	**October**	**November**	**December**
1. Obtain school consent 2. Determine functions unit learning targets 3. Write pretest 4. Develop/select 5-8 group-worthy tasks 5. Analyze each group-worthy task using Lotan (2003) and cognitive demand 6. Begin group training 7. Complete pretest	1. Complete 2-4 group-worthy tasks 2. Record observations of student discourse during group-worthy tasks 3. Complete individual follow-up tasks 4. Tally percent correct by learning target on individual tasks	1. Complete 2-4 group-worthy tasks 2. Record observations of student discourse during group-worthy tasks 3. Complete individual follow-up tasks 4. Tally percent correct by learning target on individual tasks 5. Wrap-up unit 6. Complete posttest	1. Request volunteers for student interviews 2. Obtain interview consent from students and parents	1. Conduct student interviews

Study Limitations

The purpose of this research was to address the question "How can group-worthy tasks impact student flexibility with mathematical representations when studying functions?" While the analysis shows significant gains in student understanding of representations, the pedagogical practice in the classroom is not limited to the implementation of group-worthy tasks. Other instructional practices include connecting to prior knowledge, consistent formative assessment, and high-press questioning. Additionally, there is a strong teacher-student rapport. While students are accustomed to these practices in my classroom, regardless of the unit of study, they could be considered confounding variables. Ideally, posttests would be given following only a single teacher intervention (Henning, Stone, & Kelly, 2009).

There are some factors beyond instructional approach and group-worthy tasks that may have also impacted the significant student growth. For example, some of the supplementary text problems used for the units came from the text Functions Modeling Change which adheres to the "rule of four whereby functions are represented symbolically, numerically, graphically, and verbally" (p. iii, Connally, 2006). While the students do not use this text, I use the text as a problem resource for some practice and homework. Thus, the emphasis on representations was interwoven throughout instruction and was not limited to the group-worthy tasks. While practice with representation tasks can improve student flexibility, I would argue that practice alone is insufficient; rather, "students build meaning for representations by using them in various ways. These ways include talking with other students about various representations and writing explanations about one's thinking" (p. 479, Lannin, Townsend, Armer, Green, & Schneider, 2008).

Nonetheless, students answered questions about mathematical representations outside of the group-worthy tasks.

The use of pre and posttests for measuring gains in student achievement can be problematic (Mertens, 2010). Giving a pre and posttest with the exact same question set has its advantages: the comparison in achievement is directly aligned. However, given the duration of the functions unit and students' tendencies to remember questions over a short period of time, I chose to develop a new question set for the posttest. To make the comparative analysis appropriate the test format was the same and the test questions were aligned by learning targets (Henning, Stone, & Kelly, 2009). Also, the scoring process for each test was the same.

Much like the instructional practices and the difference in pre and posttest questions, student interviews also have internal inconsistencies. Students might have responded more positively to interview questions simply to make me happy. While I attempted to address this issue by encouraging students to be as open and honest as possible, and assuring them that their honesty is most valued for the purpose of my research, a power dynamic is present in student interviews. As their teacher, students may not feel entirely at ease. Another problem with interviews is the tendency to be unintentionally leading. I carefully reviewed my recorded interactions outside of the context of the interview to see if at any point I was leading students through the interview. As I interacted with students I was cognizant of any tendency to be leading in the question-response exchange. In addition to the power dynamic, the interview participants were selected by voluntary response. Because of this, some bias is likely present in the student responses – perhaps most of the students felt very positively about my teaching and therefore eagerly volunteered. While random selection of interview participants may have eliminated the voluntary response bias, I was concerned about student availability given the timeframe to complete the study.

All in all, the findings do have some limitations due to the aforementioned concerns. However, many precautions were taken with the intent of lessening the limitations of different pre and posttest questions and student interview interactions.

Chapter III: Research Findings

Overview

Over the course of this research study I explored student flexibility with mathematical representations. In particular, I hoped to see how group-worthy tasks could impact the ease with which students tackled representation questions when studying functions. I wanted students to look at the algebraic, tabular, graphical, and descriptive representations of a function and identify the connections between each representation, as well as move from one representation to the other with ease. I also wanted students to be able to move back and forth from a process perspective to an object perspective – making use of within-representation characteristics as well as global characteristics (Maschkovich, Schoenfeld, & Arcavi, 1993).

First students completed a pretest which assessed their ability to identify and answer questions accurately with different representations. The pretest was aligned to the representations learning targets listed in Figure 2.2. Over the course of two units students then completed group-worthy tasks with a specific focus on representations within the topics we were studying in each unit. Following each group-worthy task students completed an individual task where I measured their rate of success in answering various representations questions. I found that my students' understanding improved on individual tasks following the group-worthy tasks. At the conclusion of the two units of study students took a posttest assessing the same learning targets as the pretest. Finally, a small group of students were interviewed with the intent of eliciting their perspective on the success of integrating group-worthy tasks into instruction. While I was measuring student growth in my own classroom, I also compared my students' performance in three sections of precalculus against three sections of honors precalculus (which I will simply call honors from this point on) taught without group-worthy tasks. Based on the relatively Normal distribution of scores and the large sample size, I was able to determine the statistical significance of the change in pre and posttest scores. The improvement in my precalculus classes was statistically significant ($p = 0.00000000000119$) while honors did not experience statistically significant gains ($p = 0.346$).

School Context

The school has gone through some major changes in recent years. For one, standards alignment has only recently been introduced. As curriculum director last year I led the math department through determining our philosophy, goals, and standards by course. This was a necessary process as we had opened a middle school and the scope and sequence for mathematics over the seven years of a students' mathematical progression had to be determined. Additionally, the new middle school had a more progressive direction instructionally – using Connected Mathematics Project (CMP) curriculum which focuses heavily group work and complex problem solving. Historically the math curriculum at the high school was very traditional: all classes were taught through direct instruction and the majority of student work and student assessments involved procedural understanding. In recent years the school has begun to see some changes in the high school math classes as teachers are beginning to value pedagogical approaches other than direct instruction. Also, following last year, three math teachers were let go – in part due to inflexibility with their instruction. The shift in values and the firings has resulted in greater openness to research-based practices like complex instruction. Because these changes are very recent, the students in my classes are experiencing a very different approach to learning mathematics than they have in years past.

As the philosophy, goals, and standards developed in our department, each course underwent some changes. When I began teaching at the school in 2008, the precalculus course (that now serves as the setting for the study) was much different. For one, the content was college algebra. Years prior to my hiring the course had been called Math 12. In my first year the school only allowed seniors to take the course because the material was different from a true precalculus course. The previous course was a bit of a dead-end: it was really meant to give students some additional practice with advanced algebra content so they could take precalculus the following year in college. As our department began to consider the scope and sequence more deeply, we decided the course should become a true precalculus course – open to all grade levels and serving as a step in the progression toward future coursework. This change has really only come to fruition this year. For the first time students in this course are using the same text as the honors precalculus course and the class consists of both juniors and seniors.

Research Classroom and Comparison Classroom

I am in my 13th year of teaching but my teaching experience has been in and out of the classroom. I taught 2nd-12th grade students math and SAT prep for many years at a private learning center prior to receiving my certification. After obtaining my certification I picked up a part-time position teaching high school math mid-year. In the fall of the same year I began teaching full-time at a different public high school in the same district. While teaching in the public high schools I continued to teach at the learning center. I taught full time in public school for two years. Following, I worked as a

Center Director for two different learning centers. After three years as a Director (where I continued to teach all levels of math and SAT prep) I moved back to full time classroom instruction. I have remained as a classroom teacher at the same school ever since.

My instructional approach has always focused on conceptual understanding. In my early years I taught predominantly through direct instruction. I had students look for patterns and make conjectures through my own demonstration, but not by their own investigation. I did have students work in pairs and in groups on some tasks but the tasks were typically at the *procedures without connections* level of cognitive demand. On some occasions the tasks did venture to *procedures with connections*, but less so than I would like to admit (Arbaugh & Brown, 2005). Two research based practices I successfully implemented were using high press questioning strategies (Kazemi, 1998) and having students write about their understanding frequently. Only in recent years, however, have I moved to more complex instruction techniques. Last year I trained my students to work in groups mid-year and created more group-worthy tasks in the 2nd semester. This is the first year that I began the year by training my students to work in groups and most classroom work is either completed in pairs or in groups of four. When we encounter new material, much of it is first experienced through group tasks where students conjecture and teach each other. This is a novel experience for the majority of students in my classroom.

The honors teacher, Alice, has taught for nine years. She is an exceptional mathematician and obtained her degree in Theoretical Mathematics from a highly competitive university. She started her career as an actuary and decided to make a change to teaching after five years. Alice has a passion for mathematics and works hard to ensure that students in her classroom understand mathematics deeply. She frequently makes herself available to struggling students outside of class. Alice has integrated more progressive teaching strategies in recent years – such as high press questioning, partner quizzes, and formative assessment – however, her pedagogical approach is predominantly direct instruction. I truly feel she is an exceptional teacher as she is always striving to improve and help students learn more successfully. I believe the honors classes are an appropriate comparison for the pre and posttests because the primary difference between Alice's classroom and mine is the use of group-worthy tasks.

While my classes and Alice's classes cover precalculus material from the same text, the honors level course moves through the material more quickly and works on some more procedurally complex problems (in that the students need very strong algebraic competence to get to the learning in a problem). Alice and I frequently collaborate about the content we will teach and what core content we will emphasize in each unit.

Training for Group Work

In order for students to get the most out of each group-worthy task, it was imperative that they begin with group training (Cohen, 1994). Students were arranged into groups of four. Then, over the course of two class days students completed different group training tasks. Each task addressed a different facet of working together as a group.

The first task was "Spaceship" (adapted from Cohen). Students were given a list of twelve people to take on a Spaceship to start a new civilization. Students had to work together to select seven people from the list of twelve. The objective of the task was for groups to reach consensus *before* making a decision. Additionally, students had to communicate their rationale for their decisions. Following completion of Spaceship the class discussed how they reached consensus, the importance of hearing everyone in the group, and what it meant to give a rationale for an idea.

The second task was "Puzzled Rectangles." An envelope with the pieces of four different playing cards cut into different pieces was given to each group. The group was told to pile the pieces at the center of their desks and have each group member select four pieces. The objective was to work together to complete an individual puzzle without talking and without taking pieces from other group members. The discussion that followed focused on the importance of observing what others are doing without immediately interjecting. We also talked about "pencil snatching" and how demeaning it can feel when another person takes over your thinking.

The third task was "Master Designer." Students were each given a bag of Tangrams and a cardboard stand to hide their work. One person in this task was deemed the Master Designer – they created their own design and described it to the others. The objective in this task was to replicate the Master Designer's design by asking clear and specific questions. Also, each student in the group could not see the other members' constructions. The lesson here was in using clear explanations so others could follow; moreover, many students found it was interesting to see how a peer interpreted their verbal explanations. Many students found it difficult to come up with clear and precise explanations. As with the previous two tasks, the class ended by discussing how vital it is to provide clear and detailed descriptions when working with other students in a group.

The final task was "I want your digits." This is a digit guessing game adapted from CPM Geometry which works much like the game mastermind. One student selects a three digit number and the other students in the group try to guess the three digit number using reasoning and process of elimination. This activity had group members working together to problem solve and discuss what they understood and what was unclear. Much of what we focused on in the discussion that followed was how to handle disagreements (copies of all group training tasks can be found in Appendix D).

Once the class completed all group building activities and follow-up discussions, I gave a presentation explaining the rationale for group work and the various roles and individual responsibilities students would take on during group work. We discussed group norms such as adhering to roles and being accountable to one another throughout the task.

What Made a Task Group-Worthy?

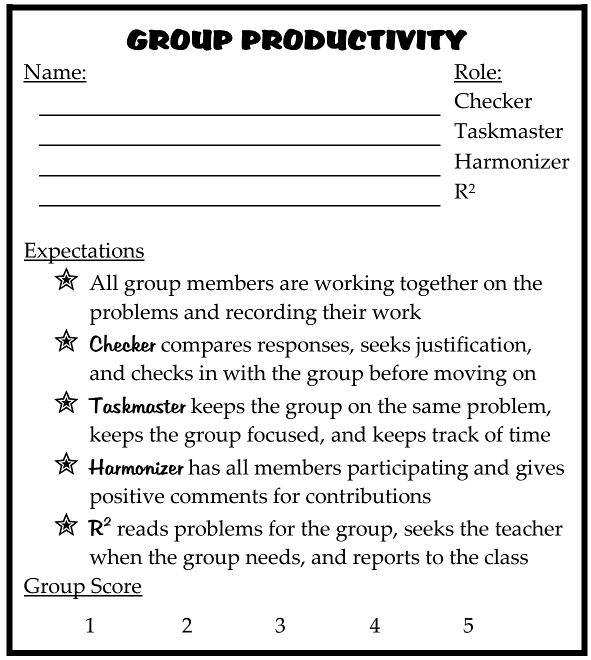
As the premise of the research question is based on group-worthy tasks, part of the research process was assessing the group-worthiness and the cognitive demand of each group task. Group-tasks were built into class time about every week. The tasks were selected based on 1) the connection to current unit material, 2) the use of various representations, 3) the level of cognitive demand, and 4) the group worthiness of the task. All of the group tasks were either at the *procedures with connections to meaning* or the *doing mathematics* levels of cognitive demand (based on the descriptors in Figure 1.3). Each task also had at least three of the five measures for group worthiness (based on the descriptors in Figure 1.2). The assessment of each task is given in Figure 3.1 (on the following page). A copy of each task can be found in Appendix E. In addition to formally developed group tasks, students spent much of their class time in pairs working through

	1.4B Activity	Ch.1A Summary	Ch. 1A Group Practice Test	1.10 Activity	1.12 Activity	1.14 Activity	1.16 Activity	Ch. 1B Group Practice Test
Group-worthy Task Characteristics								
They are open-ended and require complex problem solving.	☑				☑			
They provide students with multiple entry points to the task and multiple opportunities to show intellectual competence.	☑	☑		☑	☑	☑		☑
They deal with discipline-based, intellectually important content.	☑	☑	☑	☑	☑	☑	☑	☑
They require positive interdependence as well as individual accountability.	☑	☑	☑	☑	☑	☑	☑	☑
They include clear criteria for the evaluation of the group's product.	☑	☑	☑	☑	☑	☑	☑	☑
Levels of Cognitive Demand								
Procedures with Connections to Meaning								
Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding.		☑		☑		☑	☑	
Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas.		☑		☑		☑	☑	
Usually are represented in multiple ways. Making connections among multiple representations helps develop meaning.		☑	☑	☑		☑	☑	
Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and develop understanding.		☑	☑	☑		☑		
Doing Mathematics								
Require complex and non-algorithmic thinking – a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task.	☑				☑			☑
Require students to explore and understand the nature of mathematical concepts, processes, or relationships.	☑				☑			☑
Demand self-monitoring or self-regulation of one's own cognitive processes.			☑					☑
Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.	☑				☑			☑
Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.			☑		☑			☑

Figure 3.1: Task Assessment. The different group-worthy tasks are listed at the top and analyzed using the characteristics of group-worthy tasks and levels of cognitive demand (Arbaugh & Brown, 2005; Cohen, 1994). A ☑ indicates I assessed the task to meet the described criteria.

investigations and discussing their understanding; thus, complex instruction was not limited to group-worthy tasks.

The first group task was given on the 9th day of class. At the start of each group task students selected their roles on the group productivity sheet shown in Figure 3.2. As students worked through a group task I used the productivity sheet to allot points based on adherence to roles, focus throughout the task, discussion of understanding, and so forth. At any time a group's points could go up or down. The points earned by the end of the task were added into the students' class work grade.


GROUP PRODUCTIVITY

Name:

Role:
Checker
Taskmaster
Harmonizer
R^2

Expectations
- All group members are working together on the problems and recording their work
- Checker compares responses, seeks justification, and checks in with the group before moving on
- Taskmaster keeps the group on the same problem, keeps the group focused, and keeps track of time
- Harmonizer has all members participating and gives positive comments for contributions
- R^2 reads problems for the group, seeks the teacher when the group needs, and reports to the class

Group Score

1 2 3 4 5

Figure 3.2: Tracking sheet used for evaluating student group productivity with each group-worthy task.

How Did Group-Worthy Tasks Enhance Student Learning?

Interviews were conducted during the 1st week of December. Students had completed the posttest just about a month prior. I chose to allow for some time between the posttest and the interviews so I could find out what tasks were most memorable for students. Additionally, I had to wait until I obtained consent from each family to carry out the interviews. The five students who returned consent forms in time were Ashley, David, Paul, Lucas, and Kim.

A few themes pertaining specifically to working on the group-worthy tasks emerged with the interview questions. For one, every student had absolutely no group work experience in a prior math class. Some students had worked in pairs, but the assignment of tasks in pairs was somewhat haphazard – students worked on quizzes with the person sitting next to them without any structure, accountability, or interdependence. The second theme which emerged was the effectiveness of the group-worthy tasks. Every student who was interviewed felt emphatically that the tasks improved their learning throughout the course of the unit.

Each student had a variety of different math teachers over the course of their high school experience. Kim, a senior, was a transfer student and did not enter our school until halfway through her sophomore year. Lucas, a junior, took a math class over the summer to skip ahead. Paul, a senior, was originally in the honors class but was moved down due to poor test performance. Ashley, a senior, was at a different school her freshman year and started at our school in the beginning of sophomore year. David, a senior, had been in the same math track all years prior at our school.

When asked "How have groups or groupwork been used in your previous math classes," every student reported no formal group work being used in previous math classes. When I asked David about his experience, he said "Last year we didn't do any group work." David mentioned that some partner quizzes were given but he was consistently partnered with a very low performing, high needs, student. The partnering was without a specific intent and the tasks were not designed specifically for working together. Lucas had a similar account when he mentioned he worked in groups during

> Algebra freshman year… we only got to work in partners sometimes when we were told. This is the first year I have actually been learning with a group. Most times it's just work individually and then every once in a while there is a group project like once a year.

Each student confirmed through the interviews they had little to no experience working in groups. The experience they did have was limited to more procedural tasks. Thus my prior claim in the school context about the absence of group work is not hyperbole.

Because group work in a math class is so novel for these students, I truly thought their feelings about the effectiveness of the group-worthy tasks would be mixed – positive, neutral, or negative. Such was not the case. Every student interviewed felt strongly that group-worthy tasks positively impacted their understanding within the unit by making numerous concepts more clear and more entrenched.

Ashley's prior experiences in math were not encouraging. She experienced ongoing struggles. She was able to earn Bs but merely by completing work – in her past year in Advanced Algebra she frequently failed tests. Ashley did not feel she had a depth of understanding but she feels with the emphasis on group-worthy tasks this has changed: "[group work] helps me teach others which helps me learn and better understand."

David's prior experiences in math started poorly but improved over time. By sophomore year David was maintaining As. However, math was consistently his most difficult subject and he has had to work very hard to earn his grades. Last year he "worked really hard… did all homework and went in for extra help all the time." With discussing cognitively demanding tasks in groups, David's understanding has improved and he hasn't once come in for extra help. He had the following reflection:

> I can develop my thoughts as I say them out loud… my understanding kind of comes together as I say it… I've always been that type of person so when I don't get something being able to bounce my thinking off of someone is really helpful for me.

Paul's prior experience in math was good – he moved up to the honors track in his junior year and started this year in honors. However, after numerous struggles with poor test performance and not being able to keep up with the honors pace, he was asked to move down to my class. When asked about how the group tasks have improved his understanding, Paul said "It's helped a lot." When asked how so, he responded "it shows me different ways to do a problem… last unit when I was in a group and we were working I saw how Kelly worked through [the problem] and it like clicked in my brain." I had copies of the different group-worthy tasks from the unit and I laid them out on the desk. I told Paul I was curious if any of the tasks did anything specifically for him in his learning throughout the units. He responded with "oh, all of them did. No joke." I then asked him to select one of the activities and explain specifically what became more clear for him. He selected 1.14 Activity and said

> I remember this one really well. When we did the graphing with Antoine Dodson my group helped me figure out the flipping of the functions. I only knew how to move them around. I just think it's like better than the teacher just telling us what to do. Collaborative learning is how I best learn because I can talk it out and understand it better.

Lucas has a strong passion for math and science. Because of this he took a summer advanced algebra class so he could skip ahead to precalculus his junior year.

While math comes relatively easy for him, and he is eager, his depth of understanding is not outstanding so he was placed in my class instead of honors. Because math is easier for Lucas I was very curious about how he felt regarding the effectiveness of the group-worthy tasks. He said "I've been able to get other feedback from what other people have done and different ways of how to solve a problem… some of them easier and some of them harder." I asked him if he felt that the group held him back at all. Contrary to what I thought he might say he responded with, "we really do work together every time." I then again laid out the various group tasks we had completed and asked Lucas if any particular one had resulted in an "aha" moment. He said 1.14 Activity and pointed out that it helped differentiate between graphing with "the negatives inside versus the negative outside the function." Lucas also said the 1.12 Activity and the 1.16 Activity further solidified his understanding as he was able to discuss his thinking in his group.

Kim's experiences in math classes up to senior year were improving – she did not do well freshman year but with each new class she understood more and more how to be successful. When I asked her about her experience with the group-worthy tasks in my class, she said "I really like it because like if I don't understand something chances are someone will and if someone else doesn't then chances are someone else in the group will." I then asked if she felt it helped her understand more or if the group just copied the person that understood the material. She said "No like they explain it since we have to know it for the test. I like the activities and when we draw things on poster board because then when I'm doing my work I picture that in my head." We then talked about how the groups work well explaining things to one another and Kim brought up another point. She felt that working in groups was especially good for what she called "huge problems." I asked her what she did in the past when she had to do "huge problems." She said "Uh, just ask the teacher for help." I then asked if she felt one method – asking the teacher or working in groups – worked better than the other or if the learning was the same. Kim said

> I would have to say groups are better because if I always think back to what the teacher says then I will get mixed up but if it's different students and activities then it is easier for me to remember.

Kim recognized the group interdependence and valued this feature of a group-worthy task as it helped her remember processes for later individual tasks.

As with Paul and Lucas I laid out the different group tasks we had completed over the two units and asked Kim if anything in particular stood out, in terms of her learning, with the various tasks. Kim felt very strongly that the group practice tests were especially effective, saying

> I like the group practice tests a lot… they are my favorite. The fact that it's a test and we're working in groups it's easier to do the problems when you're not stressed out and you have people to work with when you need help. So when I take the real test I think about how we started these problems as a group.

Through the interviews I found students felt the group-worthy tasks positively impacted their learning. In fact, each student felt strongly that the tasks clarified areas of weakness, allowed them to take on more challenging material, helped solidify their understanding, and helped them recall the concepts more readily on individual tasks later on.

As students completed each group-worthy task over the two units, I followed with an individual task. The class day following the first group-worthy task, students completed an individual formative quiz (which I call a check-up). I then compared the learning target performance on the pretest with the performance on the check-up. I followed this same process with each new group task: students completed the group task, completed an individual task, then I measured their progress on learning targets corresponding to the pretest. Students completed the next group task on the 14th day of class and were given an exit ticket to measure their individual understanding. On the 16th day of class students completed a group practice test. Following, they completed an individual chapter test. Again, I measured their individual performance on the chapter test according to learning target by tabulating total percent of students completing the questions correctly. While the quizzes and tests in class are not scored using correct/incorrect, for the purpose of analysis I determined the totals in this way. The next group task was completed on the 22nd day of class. Following this activity, students completed another check-up where I totaled percent correct by learning target. The next group tasks were given on the 25th, 29th, 31st, and 32nd days of class. The posttest then fell on the 35th day of class. The progress by learning target on each individual task that followed various group tasks is shown in Figure 3.3 on page 47.

The percentages given in Figure 3.3 represent the percent of students who completed questions pertaining to the stated learning target correctly on the individual task (copies of each individual task can be found in Appendix F). As the students progressed through the unit, and completed additional group-worthy tasks, the percent of students correctly completing individual tasks increased. Not every task could address every target; thus, the --- in the table indicates the specified target was not included in the task. The table clearly illustrates an overall increase in student understanding of each learning target. Following a group-worthy task with an individual task allowed me to specifically measure individual understanding over time. Despite some small decreases in percent correct, the overall trend shows considerable growth. Some decreases with the posttest questions could be due to students putting less effort into an ungraded task. Target #11 did decline from the pretest to the posttest, but I feel this is due to the wording and open-ended nature of the question. While the pretest question was considerably straight forward, the posttest question was much more difficult for students – not because of the content but because of my choice in wording.

To further illustrate this positive trend in percent correct, Figure 3.4 shows the data using a graph for five of the unit learning targets.

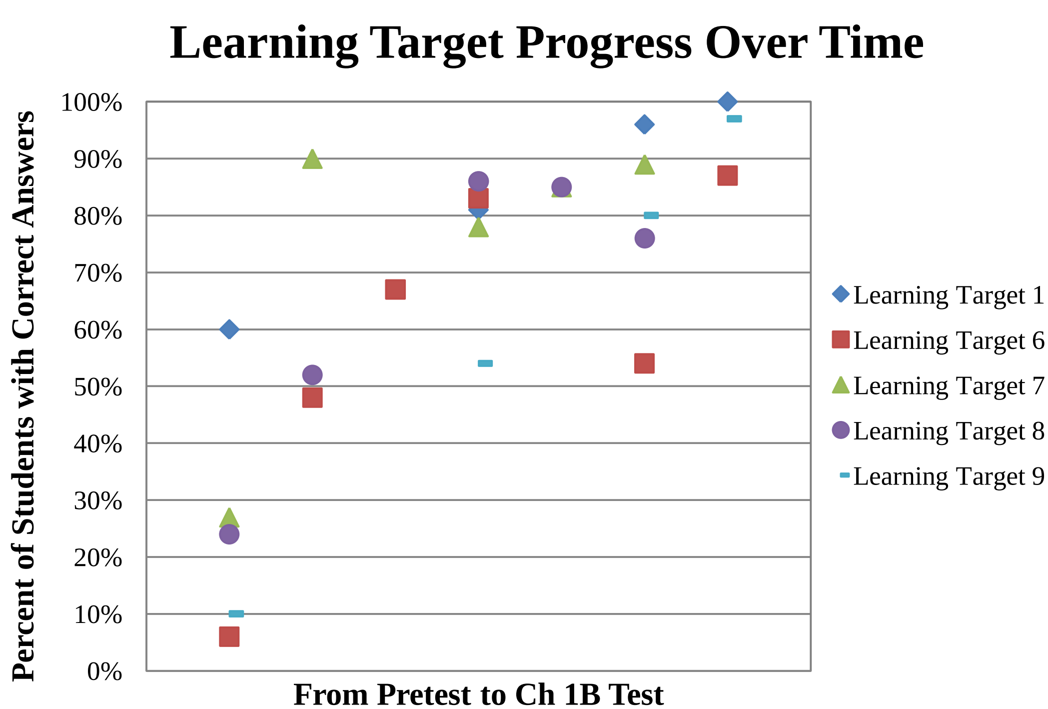


Figure 3.4: Graph of percent correct progress over the research study for five selected learning targets from Figure 3.3.

	Learning Target	**Pretest**	**1A Check-Up #2**	**Exit Task**	**Ch 1A Test**	**1B Check-Up #1**	**Posttest**	**Ch 1B Test**
1	Evaluates functions using a table	Q1a **60%**	---	---	Q5a **81%**	---	Q5a **96%**	Q3a&b **100%**
2	Identifies linear functions from a table	Q1b **18%**	---	---	Q5c **59%**	---	Q5b **80%**	---
3	Uses tables to approximate solutions	Q1c **34%**	---	---	---	---	Q5c **95%**	---
4	Uses graphs to approximate solutions	Q2b **15%**	---	---	---	---	Q4b **33%**	---
5	Identifies multiple solution strategies	Q2c **11%**	---	---	---	---	Q4c **49%**	---
6	Writes linear equations from tables	Q4 **6%**	Q1b **48%**	Q2 **67%**	Q7a&b **83%**	---	Q2 **54%**	Q7a **87%**
7	Evaluates functions using a graph	Q3a **27%**	Q2a **90%**	---	Q14a **78%**	Q1b **85%**	Q1c **89%**	---
8	Finds x values for a given function value on a graph	Q3c **24%**	Q2b **52%**	---	Q14b **86%**	Q1d **85%**	Q1b **76%**	---
9	Evaluates composite functions using a graph	Q3e **10%**	---	---	Q14d **54%**	---	Q1d **80%**	Q3d **97%**
10	Writes an equation from given function values	---	Q4 **47%**	Q1 **57%**	Q8 **81%**	---	---	---
11	Interprets a situation described by a graph	Q6a-d **90%**	---	---	---	---	Q6 **84%**	---
12	Interprets a description to create a graph	Q8 **76%**	---	---	---	---	Q7 **87%**	---
13	Translates a written description into a function	---	---	---	Q10 **73%**	---	---	Q9a **85%**
14	Connects graphs to solutions of functions	Q7a **16%**	---	---	---	Q4 **53%**	Q4a **47%**	---

Figure 3.3: Progress on individual tasks by learning target. Each unit learning target taken from Figure 2.2 is aligned to questions from individual follow-up tasks. The "Q_" represent question number. The percent represents total percent of students obtaining a correct answer. A --- indicates that learning target was not assessed on the given task.

As evidenced by the growth in percent correct by learning target, group-worthy tasks had a significant role in improving student understanding. This finding is also supported by student feedback through the formal interviews and through a written question on one of the class check-ups.

On the Chapter 1A Check-Up #2 (found in Appendix F), students were asked to reflect on which problems on their check-up were easier due to the group task. Students wrote the following comments on their check-ups: 1) "the last question was easier to understand [because] our group activity; I understand that it is just a different representation of the same data," 2)"Question 4 was easier for me because I immediately set it up in a table to find the relationship," 3) "1, 2, and 4… pretty much anything involving finding something with a function or writing functions… mostly this was because [we] wrote the equation in function notation for our table," 4) "Finding a formula was a little easier because my group explained it to me more so I understood," and 5) "Q2 was easier because I could recognize how to understand the graph and come up with equations after doing [the] group activity." Students were able to clearly link the group task to their improved learning.

As evidenced through individual measures by learning target, and through personal reports by students, group-worthy tasks did indeed enhance student learning over the course of the two function units.

What Do Students Understand About Linking Representations?

Through the administration of a pretest I was able to identify the strengths and weaknesses students held in linking and moving between mathematical representations. The pretest was administered on the 4th day of class. Students were told the pretest was intended to determine what math knowledge they had retained from previous years so I could know where to begin with instruction. I also told students they had to use the entire class period and show they were focused and trying their best. Students focused during the entire period and seemed to put their best into it. Some students took it so seriously that they wrote me notes on the test telling of their concern about the impact of their lack of understanding on their future success in the course.

The students in the honors course took the pretest following the first week of school. They were given the same explanation regarding the intent. The teacher felt students in the honors course tried their best and used their time well. In both instances students were told they would not be graded but their best effort was especially important.

The pretest scoring was correct/incorrect. Scoring in this manner allowed for less subjectivity in determining credit for a student's answer. A correct answer on a question or question part earned 2 points. For some questions that asked for justification or explanation, a score of 1 point was awarded for a correct answer without explanation. I applied the same scoring technique to my classes and the honors classes. As I graded the pretest I decided to remove one question from the totals – question 3b asked "Which is greater, $f(-2)$ or $g(1)$?" There was no way for me to determine if students obtained the correct answer by guessing. I then totaled each student's points and determined a percent correct.

As I had anticipated, students in the honors class did considerably better on the pretest than students in my precalculus class. Figure 3.5 gives a side-by-side histogram with totals by individual percent scores. As one can see from the graph, the scores in my precalculus class were on the lower end with some skew to the right. The skew to the right can be attributed to some students taking my class instead of honors for a better grade despite a stronger ability level. The honors precalculus course has some skew to the left which can be attributed to students seeking an honors course for their transcript even when they may not be strong math students.

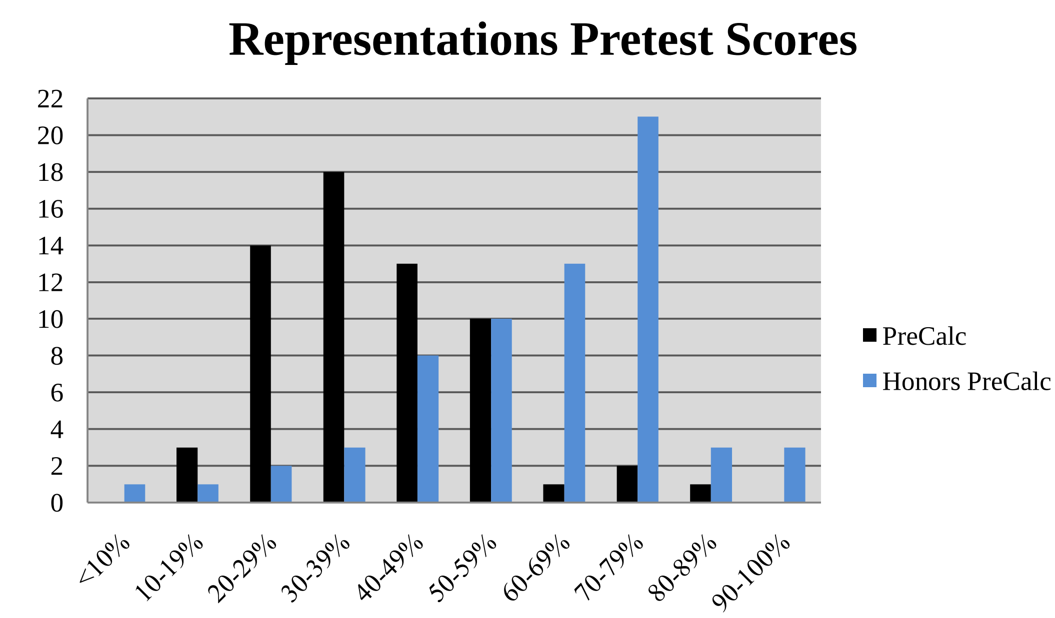


Figure 3.5: Pretest scores by class. The x-axis represents earned score while the y-axis is a total count of students who earned a score in the given range.

A number summary for each class can be seen in figure 3.6. The difference in mean score on the pretest is quite substantial. To determine possible outliers I used the 1.5xIQR rule. While the median is resistant to outliers, the comparison with posttest performance later will use the mean; thus, I wanted to determine if any outliers existed so that I might include an adjusted mean down the road.

	Precalculus	Honors
median	38%	68%
mean	39%	62%
std dev	14%	18%
IQR	18%	24%
outliers	1 > 74%	2 < 14%

Figure 3.6: Number summary using earned score by class.

There were consistent problem areas in the pretest. Firstly, students in my class made little to no connection between equations and graphs – they looked at each as separate objects. Very few students recognized multiple means to find solutions (such as algebraic, graphical, or tabular). Those students that could connect a function to a graph frequently mixed up finding a function value for a given x value versus finding an x value for a given function value. Students also relied heavily on algebraic approaches even when a different method was more efficient. For example, in problem 2, when students were given an equation and its corresponding graph, then asked to find an approximate value of y when x is 3, they rarely used the graph to identify the point; rather, they set x to 3 and solved for y. See Figure 3.7 for a sample of student work on this question.

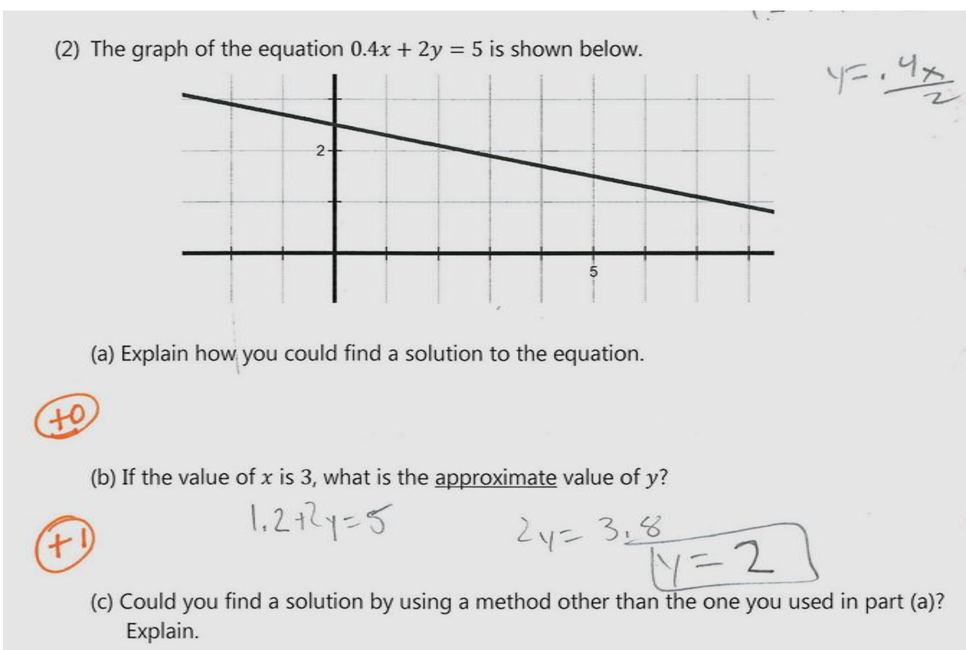


Figure 3.7: Sample student response for pretest question from the Precalculus class.

The honors students had considerably greater strengths on the pretest. They were much more successful in evaluating functions from a graph, in finding solutions for a function, in determining a trend and predicting a future value, and in writing a linear equation for a table of values. However, the honors students relied heavily on algebraic approaches: they found intercepts when asked for a solution for the graph of a function and used the slope-intercept process for finding the linear equation for a table.

While the honors students did better with the different representation tasks, they still demonstrated some areas of low flexibility when they relied heavily on less efficient algebraic processes. This same phenomena has been observed in previous research studies (Even, 1998; Hitt, 1998; Knuth(a), 2000). However, the results of the pretest do confirm students with stronger mathematical understanding have greater flexibility with representations.

The pretest results confirmed that my precalculus students had a significant deficit in their ability to link and move between representations. However, as the classes moved through the two function units I started to see not only gains by learning target, but evidence specifically tied to representational flexibility. Over the course of the research study I made notes in a research journal. As students worked through different tasks, I jotted down my observations. These were informal notes of what stood out as improvements in use of representations and flexibility.

One of the first improvements I noticed followed the first group-worthy task on patterns. Students started consistently making tables to find patterns. For example, on 1A Check-Up #2, when told to find a formula for the linear function $f(x)$ given $f(0) = 3$ and $f(4) = -1$, many students put the values in a table, found the slope by looking at how much $f(x)$ changed from $f(0)$ to $f(4)$, and then proceeded to write the equation. While some might argue that using the point-slope form to find an equation would be more appropriate, the use of a table showed considerable depth of understanding and flexibility between the representations – students looked at the function representation and the table as connected!

A few days later I saw that students were immediately translating function values into coordinates. The connection to a coordinate and a point on a graph seemed painfully obvious to them. So much so, that when I prompted them to write function values in multiple ways, students were almost annoyed as if I was insulting their intelligence by asking them. Additionally, any question asking students to translate a description into a linear function was especially easy. Alice, the honors teacher remarked about how astutely my students were using function notation and talking about different representations (I teach one period in her classroom during her planning period).

When I gave students Activity 1.12 I was honestly concerned with the difficulty of some parts of the task. In particular, I was unsure if they could write the transformation of one function in terms of another. To my surprise, the majority of students had little problems with the task and were able to understand the horizontal and vertical transformation of functions. Finally, with Activity 1.14, students swiftly wrote transformations using function notation from graphs. When they had to prove their transformation rule using a point on each figure, they knew exactly what to do and seemed to look at all features of the task (graphs, coordinates on graphs, and function representation) as an interconnected system rather than as separate objects.

One final observation was in dealing with questions where students were given a function and an output and asked to find a corresponding input. For instance, an in class practice task had the following problem: Let $f(x) = \left(\frac{x}{2}\right)^3 + 2$, find when $f(x) = -6$. In the past, students would approach this problem by 1) incorrectly substituting -6 for x, 2) setting $f(x)$ equal to -6 and then feeling unsure as to how to proceed for solving, 3) skipping the problem, or 4) solving it correctly algebraically (though small numbers of students were able to do so accurately). What happened over the two units was a pleasant surprise – students started thinking conceptually about this type of problem and looked for what value for x, as in input, would result in the desired output; thus, they would say $\left(\frac{x}{2}\right)^3$ has to be -8 in order to end up with -6 after adding 2. For $\left(\frac{x}{2}\right)^3$ to simplify to -8, the number in parenthesis would have to simplify to -2, and for that to happen x would have to be -4. Students were actually having these types of conversations as they reasoned through the problems. This showed up on the Ch. 1B Group Practice Test when students were given a height function for a projectile ($d(t) = -16t^2 + 64$) and asked to determine the time when the object hit the ground. Students were reasoning in their groups about what value would make the height zero and then determined the appropriate time. Most groups were sketching out approximate graphs and considering the path of the projectile. Quite a few students used the same approach on the Chapter 1B Test where the problem was if $\mathrm{g}(x) = x^2 - 10$, when does $\mathrm{g}(x) = 6$? Truly the majority of students simply wrote $g(4) = 6$ as their answer. While this does not give both possible solutions that one might obtain algebraically (if done correctly), it at least demonstrated that students were thinking about the algebraic representation in ways that I had not seen in previous years. In addition to using a more conceptual approach, students actually wrote their answer in function notation. Students were able to demonstrate this growth in understanding in their posttest performance as well.

The posttest was administered on the 35th day of class. The questions on the posttest were very similar to the pretest questions. The assessed learning targets were the same. Many of the questions were very similar but involved a different graph or table so as to avoid an exact replica of the pretest questions. I wanted to see how students would apply their improved understanding with novel problems. While I retained their pretests, and did not review the answers with them, many students discussed their responses to each question with each other. As a result, I did not want to use the exact same questions on the posttest. The word problems addressed the same targets and were grounded in the same context, but involved very different descriptions. The honors students completed the posttest the same week. Again, as with the pretest and each individual task throughout the study, students were scored based on correct/incorrect responses with 2 points awarded for a correct answer. Students did earn 1 point if they gave a correct solution without an explanation or justification. While awarding the same points for each question does not reflect the difference in value of each question, it allows for ease of comparison when looking at each task. One posttest question resulted in different interpretations than I had intended (Q7). Many students created bar graphs instead of line graphs. Hence, after discussion with the honors teacher, I decided to award credit based on a reasonable interpretation of the question with a labeled graph. This question resulted in the one target in Figure 3.3 that decreased in percentage from the pretest to the posttest.

The score results of the posttest are displayed in Figure 3.8. This side-by-side histogram uses the same percent earned increments as in Figure 3.5 with the pretest performance. There are many noteworthy changes here. For one, the difference in performance between the precalculus and the honors course is far less significant. What's more, the peaks in each class are at adjacent increments whereas in the pretest the peaks were four increments apart. In examining the pretest with the honors classes, the peak occurred at the 70-79% range where in the posttest the peak occurred at the 60-69% range. In my precalculus classes the pretest peak occurred at the 30-39% range where in the posttest the peak occurred at the 50-59% range. Not only did students in my precalculus class make major improvements, but the gap between achievement of my students and the honors students reduced significantly.

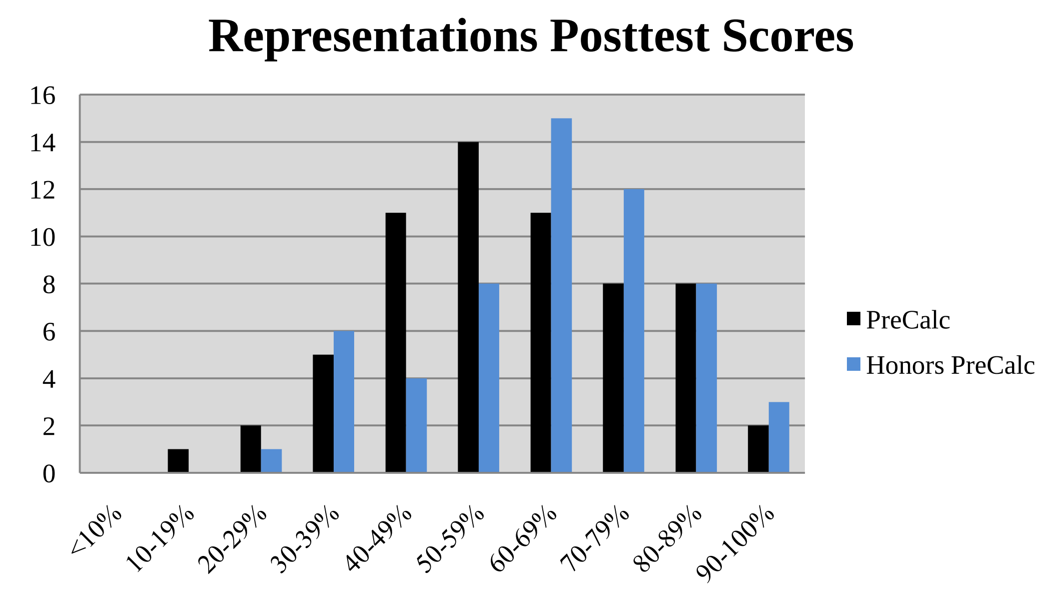


Figure 3.8: Posttest results by class. The x-axis represents earned score while the y-axis is a total count of students who earned a score in the given range.

To further illustrate the difference in pre and posttest results for each class, a comparative number summary is given in Figure 3.9. The precalculus class went from an initial mean of 39% to a posttest mean of 60%. The honors class went from an initial mean of 62% to a posttest mean of 65%. The standard deviation and the IQR are both a means to measure the spread of data. While the gains in the precalculus posttest are significant, one should note that the spread of data increased from the pretest to the posttest. This means there is greater variability in the scores and students are performing at a wider range of ability levels.

	Precalculus		Honors	
	Pretest	Posttest	Pretest	Posttest
median	38%	58%	68%	67%
mean	39%	60%	62%	65%
std dev	14%	18%	18%	17%
IQR	18%	27%	24%	22%
outliers	1	0	2	1

Figure 3.9: Comparative number summary using earned pre and posttest score by class.

To test the significance of the posttest gains, I performed a matched pairs t-test. I felt it was important to compare each student against his or herself rather than oversimplify the comparison by only looking at the mean of the entire class with the posttest. Because the data was close to Normal and the sample was large, a test of statistical significance is appropriate. Each students' pre and posttest scores were matched. The mean difference for the matched pairs was 21% with a standard deviation in the mean difference of 18%. The t statistic obtained was $t(58) = 8.885$ with $p = 0.00000000000119$. With such a small p value there is almost no chance (1 in 1 trillion) that the gains in posttest results happened by chance.

Students demonstrated their improved flexibility with representations through their approach to answering posttest questions. On a question where students were asked to find a formula for a table, the honors students used a point-slope process and substituted values to find the intercept. On the other hand, the precalculus students from my class identified the rate of change and the intercept directly from the table and accurately wrote the linear function formula.

Also, on a question asking students to interpret a graph for a bike ride, many honors students had the same misconception. They interpreted a negative slope on the graph as a downhill motion. And, at the point on the graph where the bike rider is stationary, many honors students identified this as a "constant rate of change" and stated that at this point the bike rider is on a flat surface riding the same rate. No students in my precalculus classes had this same misconception. This may be due to the honors course's emphasis on increasing, decreasing, and constant intervals, but it nonetheless demonstrates a lack of conceptual understanding.

Thus, not only do the total earned scores demonstrate an improved understanding of, and flexibility with, representations but also the students' responses and approaches to solving each problem on the test substantiate this finding.

Can Students Identify Strengths and Weaknesses of Different Representations?

In addition to flexibility in moving between representations, a student with depth of representational understanding should also understand when it is most appropriate to use, for example, a graphical instead of an algebraic approach to solving a problem. The honors students continued to demonstrate a lack of flexibility in their responses to some questions. As with the pretest, on the posttest the honors students relied heavily on algebraic processes. They were unable to identify the strength of the graphical approach. When given the function $g(x) = -x^3 - 2x^2$ and its corresponding graph, students found the zeros of the function algebraically in order to "give a solution to the function." Additionally, rather than state the solution of "when $y = 0$, $x = 0$ or $x = -1$" students instead just gave the answer of "$x = 0$ or -1." In doing so they identified the graph and equation as separate, disconnected, objects and did not recognize how the x values obtained represented solutions. They answered the question as if the graph was not even present. For an example of a typical honors precalculus students' work on this question, see Figure 3.10.

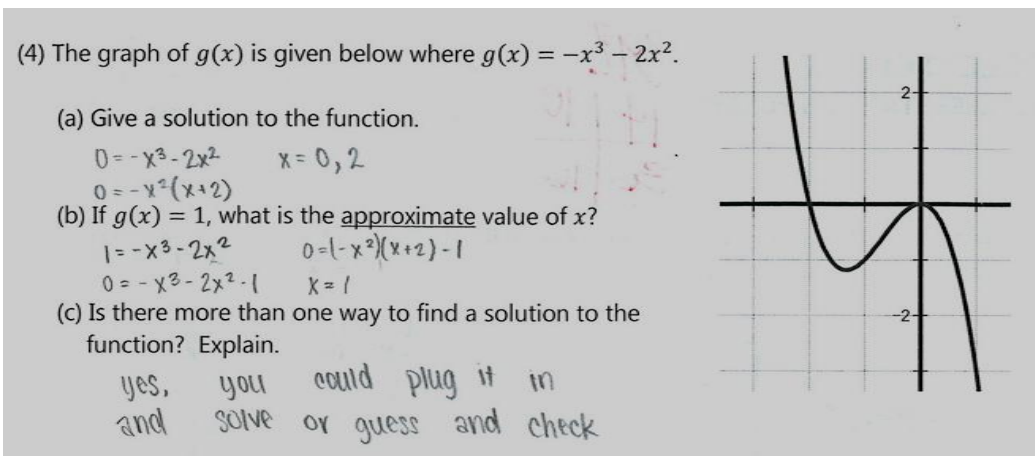
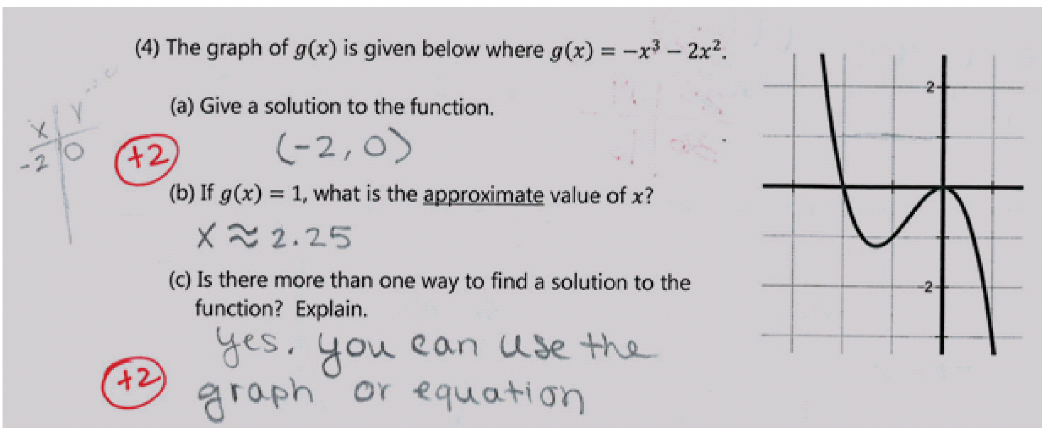


Figure 3.10: Honors student sample response on posttest.

Alternatively, students that successfully answered this question in my precalculus classes simply identified a coordinate on the graph as a solution. On this same question, when students were asked to identify an approximate value of x for a given solution value, many honors students solved the equation rather than identifying the appropriate location on the graph (see Figure 3.10 part b). While the algebraic approach can be an accurate means to obtain a solution, most students couldn't get an actual answer because they did not yet know how to solve that type of polynomial equation. A typical example of the same question from Figure 3.10 for a student in my precalculus class is shown in Figure 3.11. Clearly in this example the student is identifying the strength of using the graphical representation. Additionally, the student shows a connection to another representation by showing a small table for the point identified.



(4) The graph of $g(x)$ is given below where $g(x) = -x^3 - 2x^2$.

(a) Give a solution to the function.

+2 (-2, 0)

(b) If $g(x) = 1$, what is the approximate value of x?

x ≈ 2.25

(c) Is there more than one way to find a solution to the function? Explain.

+2 yes. you can use the graph or equation

Figure 3.11: Precalculus student sample response on posttest.

I sought to further confirm student improvement in this area through the interviews. One of the final questions asked of students in the interview pertained specifically to strengths and weaknesses of different representations. When I asked Ashley if there was something she could easily find on a graph that she couldn't easily find from an equation she said "Yeah, the slope, the intercepts." When I then asked her about the weakness of using only an equation she said "You can't like visualize it… you can't understanding if it's going up or down and can't figure out the deeper meaning." When I asked David the same questions he responded with

> Hmm… well personally I am a visual learner so graphs are helpful for me to see what's going on… when I'm working on problems on my own I'll even just graph it on my own so I can get an idea of what is happening… so that's helpful for me… negatively a sketch is sometimes not that accurate… I have to do the math to find the intercept or something. For me looking at an equation isn't really visually helpful but it helps me get exact answers.

Lucas looked at me oddly when I asked the question. I smiled. He responded with

> The graph is usually a product of the equation. You graph what the equation is telling you. They are connected to each other… [the graph] puts visuals so you can see what the equation is actually doing… back in freshman year just looking at an equation it didn't make sense at all… looking at a graph helps.

I found it interesting that Lucas said the weakness of just seeing an equation is "some equations to people look like uh I'm not going to be able to do that" as if the algebraic representation was so meaningless that students just gave up at the sight of it.

Students clearly made a connection between the representation of a problem using a graph and an equation. Furthermore, the fact that they were able to determine strengths and weaknesses of each showed a true grasp of flexibility between representations.

Summary

Group work and group-worthy tasks had a clear positive impact on student flexibility with representations over the course of the two function units. Gains were demonstrated through multiple data sources – pre and posttest results, individual tasks, classroom observations, and individual student interviews. Additionally, the classes that served as a comparison group did not show statistically significant gains while students in my classes did. Group work has been shown to improve student understanding and reduce achievement differences (Blatchford, 2003; Boaler & Staples, 2008; Cohen, 1994; Cohen & Lotan, 1995; Prichard, Stratford, & Bizo, 2006; Slavin, 1991; Stein & Henningsen, 1997). This study showed group-worthy tasks could successfully be targeted at improving a specific area of weakness amongst high school math students. While the group work and carefully crafted group-worthy tasks clearly helped in improving student understanding, there were other factors at work. My instructional approach includes additional research-based strategies like high press questioning, connecting to prior knowledge, formative assessment, and writing about understanding. These serve as confounding variables as they cannot be separated from the addition of the group-worthy tasks. However, in previous years, where all of the aforementioned practices except group-worthy tasks were in use, student weaknesses with mathematical representations persisted. Thus, this does provide evidence that group-worthy tasks can positively impact student flexibility with mathematical representations when studying functions.

Chapter IV: Conclusion

Connecting Study Findings

This study explored how the use of group-worthy tasks might impact student flexibility with mathematical representations when studying functions. Numerous previous research studies have shown the power of group work and group-worthy tasks with the intent of reaching greater numbers of students in heterogeneous classrooms, closing achievement gaps, and building stronger conceptual understanding (Boaler, 2008; Boaler & Staples, 2008; Cohen, 1994; Cohen & Lotan, 1995; Gillies, 2003; Lotan, 2003; Lotan, Cohen, & Holthuis, 1994; Slavin, 1991). However, little research has explored how the use of group-worthy tasks could target a specific area of mathematical understanding.

Over the past year I had witnessed the power of group work with improving depth of understanding and concept retention in my classroom; however, I consistently observed student weakness with representational flexibility. Thus, I decided to pursue the idea of writing group tasks with particular emphasis on moving between and interpreting representations.

As indicated through pre and posttest results, classroom observations, and student interviews, students in this research study improved their flexibility with mathematical representations. In the pretest students struggled to identify function values and solutions from a graph, they attempted to apply less efficient algebraic methods to solve problems (yet did so unsuccessfully), and, while at times provided correct responses, they could not justify their answers. As students completed the two function units they showed consistent improvements with each of the unit learning targets in their individual tasks and in their discussions with group members. By the time students completed the posttest they were able to demonstrate statistically significant improvements in all areas of weakness from the pretest. This study illustrates how the power of group-worthy tasks can improve a specific area of mathematical weakness; namely, flexibility with mathematical representations.

The study pretest results demonstrated students' weakness with linking representations and their strong reliance on algebraic methods. This is not an unusual finding as research has indicated students' persitant reliance on less efficient algebraic methods to solve problems while concurrently struggling to link the equation, table, and graph representations of functions (Coulombe & Berenson, 2001; Even, 1998; Gagatsis & Shiakalli, 2004; Goldin & Shteingold, 2001; Hitt, 1998; Kaput, 1987; Knuth, 2000a; Knuth, 2000b; Maschkovich, Schoenfeld, & Arcavi, 1993; Meij & Jong, 2003; Rider, 2007). Additionally, students in the honors course performed significantly better on the pretest than those in the precalculus course. This seems to further substantiate the idea that students with stronger math understanding have greater representational flexibility (NCTM, 2009). However, through the course of the research study students were able to close the gap between themselves and the higher math achievers in the honors course.

Many researchers have acknowledged student weakness with mathematical representations and, as a result, have provided suggestions for teachers and curriculum to improve this particular problem area. Knuth (2000) suggests encouraging students to present solutions with multiple representations. Aspinwall (2007) gives the recommendation that instruction should include two components: an emphasis on situations in which one representation has an advantage over another and a discussion of what it means to translate from one representation to another. Gagatsis and Shiakalli (2004) propose "instruction should include all modes of representation in the translation tasks because each representation has its own characteristics and poses different challenges for students" (p. 655). Rider's research (2007) led her to conclude that teachers should present new concepts using different representations without preference; she also felt strongly that assessment questions should reflect this emphasis on moving between representations. Goldin and Shteingold (2001) explored students' external and internal representation systems. After summarizing the different stages of understanding representations they purport "in teaching every mathematical topic, we should see the development of strong, flexible internal systems of representation in each student as the essential goal" (p. 19). Friedlander and Tabach (2001) explored how to promote verbal, numerical, graphical, and algebraic representations in algebra. Their instructional recommendation is to improve understanding of representations through targeted questioning and encouraging the use of various representations when presenting the solution to a problem. Over the course of this research study all of these instructional recommendations were put to use through the problem development in each group-worthy task. This research study goes one step further – it takes the aforementioned recommendations, puts them to use specifically through group-worthy tasks, and measures student growth over time. What has been published until now only recognizes and provides suggestions as to how to improve the problem of inflexibility with representations. With the addition of this research study, however, there is clear evidence of how to effectively address student weaknesses.

Implications of Findings to Practice

As I continue to develop my craft as a teacher I will strive to find/modify/develop group-worthy tasks for use in my classroom. Students are truly willing to work in groups; and as indicated through their interview responses, students want – specifically in a math classroom – to discuss their understanding in groups. When students are trained to work together, held accountable to their roles, and when the task is both group-worthy and cognitively demanding, the learning becomes more than I could have ever dreamed (had I simply demonstrated a concept at the front of the room). The most challenging part of incorporating worthwhile groupwork is developing the tasks. However, I find application problems or problem solving tasks in most textbooks can become group-worthy tasks through slight modifications and removal of scaffolding. This makes the process of developing tasks a bit more attainable.

As I started implementing group work in my own classroom to a greater extent, letting go of the "sage on the stage" role was uneasy. I started as a math teacher because I felt I had a gift for explaining mathematical concepts in ways that anyone could understand. I still allow myself some opportunity to do this. However, the more times a student can explain and discuss their understanding with a peer, the more likely a student is to learn a concept deeply and with better retention. As I share the responsibility for learning with my students, they become more successful in mathematics – a lesson I remind myself of frequently when I have a desire to fall back on preaching math to my students.

A further question that occurred to me following the completion of the research was: to what extent had students internalized their understanding? While students had performed strongly on the posttest, I was unsure of how they would extend this understanding to different contexts as we progressed to new units. Following the conclusion of the formal data collection we moved on to polynomial, rational and exponential functions. With each new unit I am impressed by the representational knowledge students demonstrate. On a recent group test I asked students to graph two simultaneous equations by hand ($f(x) = -\frac{1}{2}x + 3$ and $g(x) = 3^{x-1} - 1$) and then I asked for what values of x is $f(x) > g(x)$? I hesitantly added this as a "stretch" question because I doubted students could determine the correct response but I felt it was worthwhile to see how they grappled with it. In my experience, this is often the type of question that lower level students in a college precalculus or calculus course have trouble making sense of. After grading the group practice test I found all but one group in all three of my classes answered this question correctly.

I also witnessed the longevity of improved representational flexibility while teaching a recent SAT prep class. One of the juniors from my precalculus class is also taking my SAT prep class. When we got to the lesson on function understanding, which is always difficult for students, I put the student from my class on the spot. The questions that day dealt with graphical representations of functions and finding function values for inputs or finding input values which corresponded to specified function values. These types of questions generally appear in the "hard" section of the math SAT. I hesitantly called on the student from my precalculus class to explain how she interprets these questions. I was relieved when she clearly explained to the class how to connect the function notation and graphical representation. She is a much lower math achiever and was beaming with pride after many students in the class responded with "ooohhhhh" and "wow, how did you know how to do that?" Thus, while I was initially concerned about students' ability to retain their improved understanding, I am witnessing consistent evidence of strong internalization.

Some additional areas for me to explore in my practice include working specifically with resistant group members and monitoring dialogue during group work. While the group work has been successful overall, there are still a handful of students who are resistant. A next step for my research with group work and group-worthy tasks could be working with resistant students specifically and exploring the rationale behind their hesitation. Some of them could simply still be unsure of how to ask questions in groups or how to share understanding. Perhaps they could benefit from additional training. Whatever their reason, these students would gain more from the group work if I was to work with them exclusively and provide strategies for getting more out of each task.

Now that I have developed an understanding of how to implement group work and group-worthy effectively, I would like to analyze how students work together more closely. As the focus over this research project was more about the task, I think the natural next step is to explore student dialogue within the task. I would like to videotape or voice-record student interactions. While I am always roaming and addressing questions brought up by individual groups, I am sure there is much going on within each group that I do not catch. For instance, were students really justifying their thinking and clarifying understanding for one another when I wasn't standing near them? Was the contribution of each group member relatively equal? Did mixed gender or mixed-ability groups have equal sharing? Did group members listen to one another's ideas? Listening or watching for these types of interactions would be helpful in my ability to develop more effective group participants; which, in turn, would result in better learning. Cohen (1994) recommends systematic interaction scoring where an observer tracks group interactions. Furthermore, she suggests using student questionnaires to determine distribution of communication and work within the group. These types of analyses would be a fruitful next step as I work to improve the utility of group work in my classroom.

Areas for Research Outside My Classroom

Students in this study made noteworthy improvements in their flexibility with mathematical representations. However, it is difficult to pinpoint one cause for student gains. As I have stated previously, my classroom practice involves many research-based practices which have proven successful with struggling students in years past. As is the case in most classrooms, there is usually more than one specific pedagogical technique at work at any one time. Thus, with confounding variables at play, it is not appropriate to conclude a direct cause-effect relationship between the treatment and the posttest improvement. However, given the comparison of the honors group, the group-worthy tasks did have a positive impact on student learning specific to flexibility with representations. It would be appropriate to try a similar treatment in a different, perhaps historically traditional, classroom to see if similar improvements would result.

An additional area of possible research would be use of group-worthy tasks to target other specific deficit areas in mathematics. This research project was unique in that prior research has not used group-worthy tasks to target one overarching theme. It would be interesting to see how the same approach could be used for other thematic problem areas in mathematics.

Finally, while I argue that the subjects in this study are average to lower average math achievers, they are predominantly white and middle to upper class. It would be appropriate to try this same type of treatment in a setting where other factors are at play. Lewis (2008) points out that economic, social, and cultural capital impact each student in their ability to learn in a classroom. These factors have a significant influence in an urban public school setting and, while present, are far less of a factor in my classroom.

Comments

As other teachers look at implementing this same practice in their own classroom I think it is important to understand not all work in my classroom is completed in groups. I still do some demonstration and students also frequently work in pairs. I believe balance is important and I do not believe *every* concept is appropriate for group work. Additionally, I think it is vital to bring the class together following a group task and summarize learning, discuss questions, present solutions, and then look at how student learning can be applied to a different context. Group work at its best has an objective and a means to assess student understanding following completion of the task.

Implementing group work does feel awkward at first. However, with persistence, collaboration, and training any teacher can integrate group work. This, in turn, will undoubtedly result in positive achievement gains for even the lowest of students.

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Appendix A

Research Pretest

PRECALCULUS
REPRESENTATIONS PRETEST[1]

(1) Use the table given below to answer each question that follows.

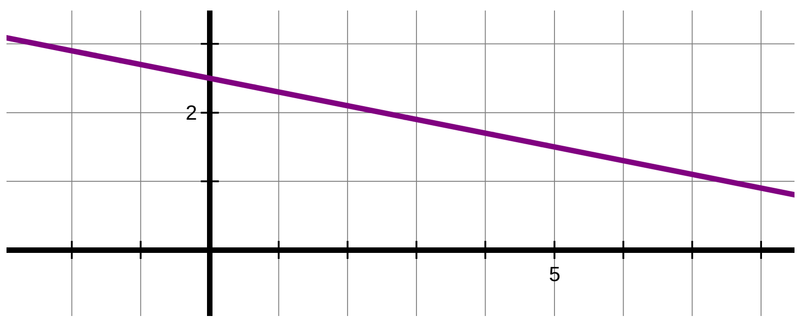
x	-2	-1	0	1	2	3	4	5
$f(x)$	1	1.25	3	5	7.6	12	18.5	30

(a) What is $f(2)$?

(b) Would you describe the data in the table as linear? Why or why not?

(c) Explain how you could approximate $f(6)$.

(2) The graph of the equation $0.4x + 2y = 5$ is shown below.

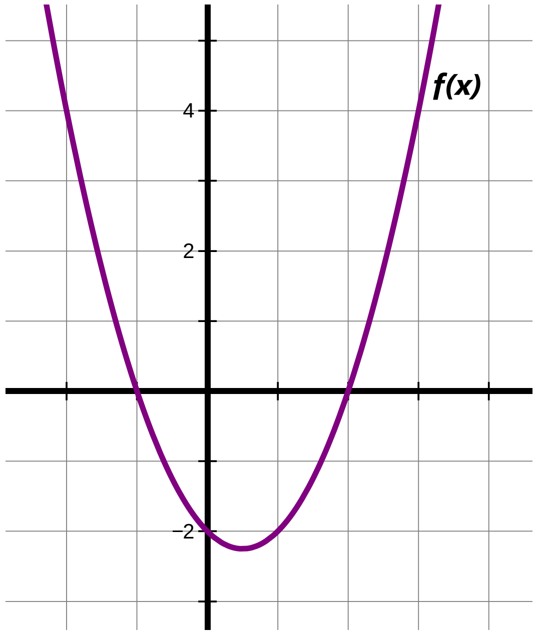


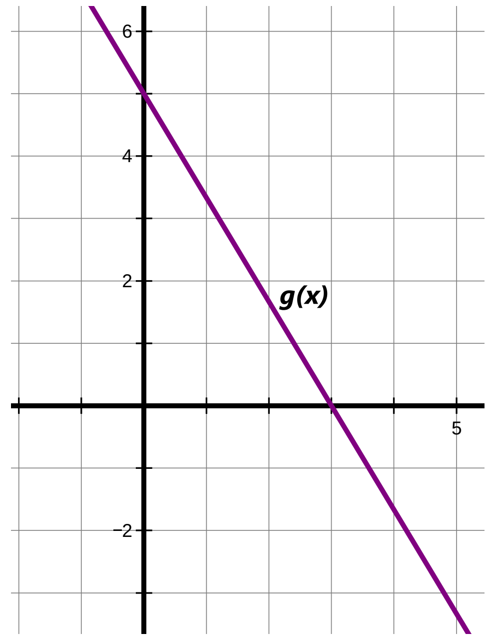
(a) Explain how you could find a solution to the equation.

(b) If the value of x is 3, what is the approximate value of y?

(c) Could you find a solution by using a method other than the one you used in part (a)? Explain.

(3) Given the function $f(x)$ on the left and $g(x)$ on the right, answer each of the following questions.





(a) Find $g(0)$.

~~(b) Which is greater, $f(-2)$ or $g(1)$?~~

(c) When does $f(x) = 0$?

(d) Find the product of $f(1)$ and $g(3)$.

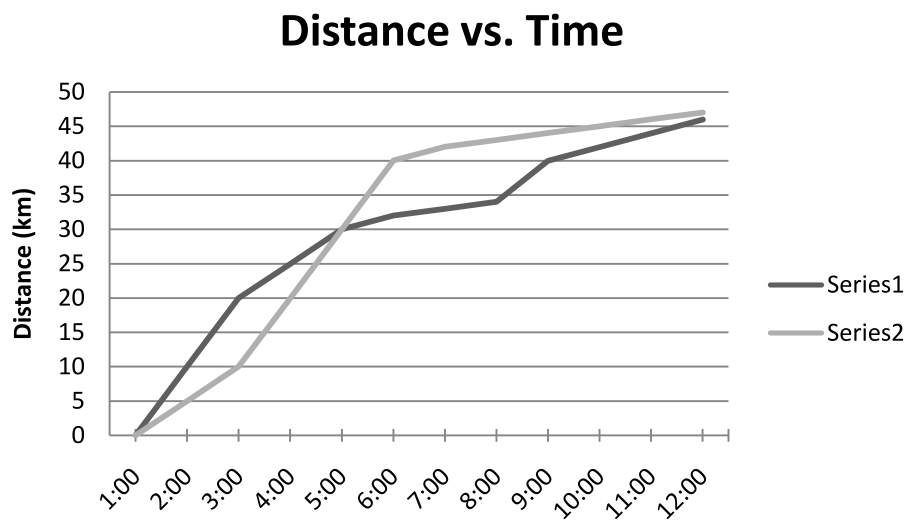
(e) Find $f\big(g(3)\big)$.

(4) Find an equation for $f(x)$ using the values in the table below.

x	$f(x)$
-1	-1
0	2
1	5
2	8
3	11

(5) If you substitute 3 in for x in the function $y = mx + b$ you get a negative value for y. If you substitute -2 in for x in the same function, you get a positive value for y. What does this tell you about the slope of the function? Explain your answer.

(6) The graph below shows the distances traveled by Ms. Petersen (Series1) and Ms. Maletta (Series2) over a period of time. Use the graph to answer the questions that follow.



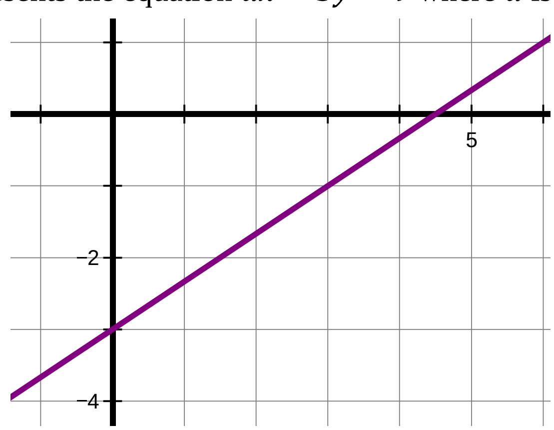
(a) For what times had Ms. Petersen traveled further than Ms. Maletta?

(b) At what time did Ms. Petersen travel the fastest? Explain.

(c) Who traveled the fastest from 3:00 to 4:30? Explain.

(d) Who traveled the greatest distance from 5:00 to 12:00? Explain.

(7) The graph below represents the equation $ax - 3y = 9$ where a is a constant.



(a) Can you find a solution to the equation without knowing the value of a? Explain.

(b) Explain how you could find the value of a.

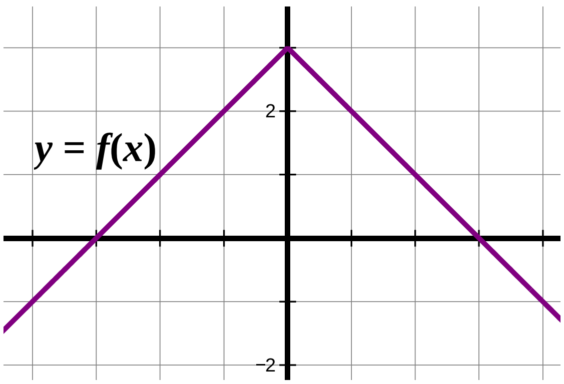
(8) A person's blood sugar level at a particular time of the day is partially determined by the time of the most recent meal. After a meal, blood sugar level increases rapidly, then slowly comes back down to a normal level. Sketch a person's blood sugar level as a function of time over the course of a day. Label the axes to indicate normal blood sugar level and the time of each meal.

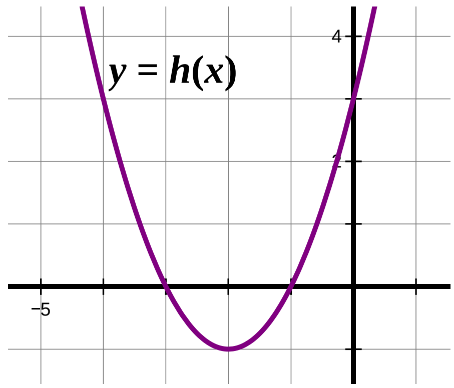
Appendix B

Research Posttest

PRECALCULUS
REPRESENTATIONS POSTTEST[2]

(1) Given the function $f(x)$ on the left and $h(x)$ on the right, answer each of the following questions.





(a) Find $f(0)$.

(b) When does $h(x) = 0$?

(c) What is $f(1) \cdot h(0)$?

(d) Find $f\big(h(-2)\big)$.

(2) Find an equation for $f(x)$ using the values in the table below.

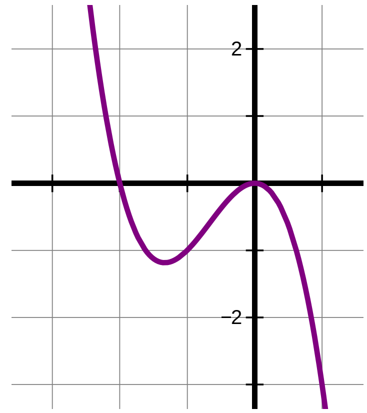
x	-3	-2	-1	0	1
$f(x)$	4	6	8	10	12

(3) Given $f(x) = mx + b$, how will the slope and intercept compare in the transformed function $f(-x)$?

(4) The graph of $g(x)$ is given below where $g(x) = -x^3 - 2x^2$.
(a) Give a solution to the function.

(b) If $g(x) = 1$, what is the <u>approximate</u> value of x?

(c) Is there more than one way to find a solution to the function? Explain.



(5) Use the table given below to answer each question that follows.

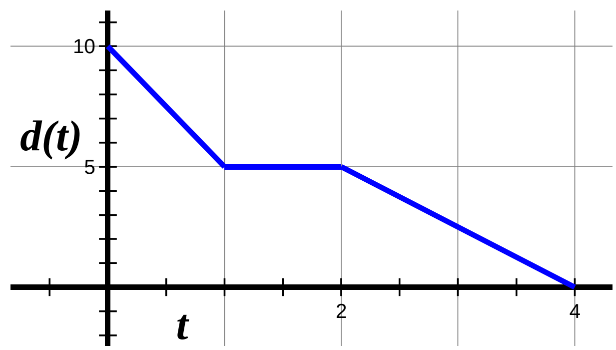
x	-2	-1	0	1	2	3	4	5
$f(x)$	-8	-5	-2	1	4	7	10	13

(a) What is $f(-1)$?

(b) Would you describe the data in the table as linear? Why or why not?

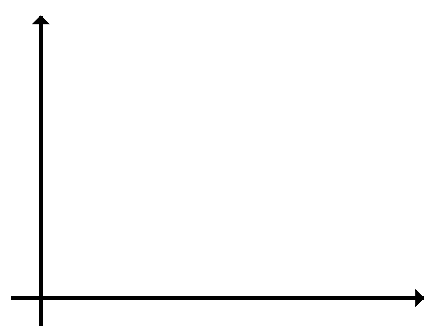
(c) Explain how you could approximate $f(10)$.

(6) The following graph represents a bike ride where $d(t)$ is the distance in miles and t is the time in hours. Write a story that corresponds to the graph.



(7) Visiting Starbucks daily for your morning coffee fix can get very expensive. In the month of September Ms. Maletta visited Starbucks every day and purchased the same coffee drink each day. During the month of October Ms. Maletta started drinking a more expensive drink every day. Then, during November, Ms. Maletta is going to make her own coffee and not visit Starbucks at all.

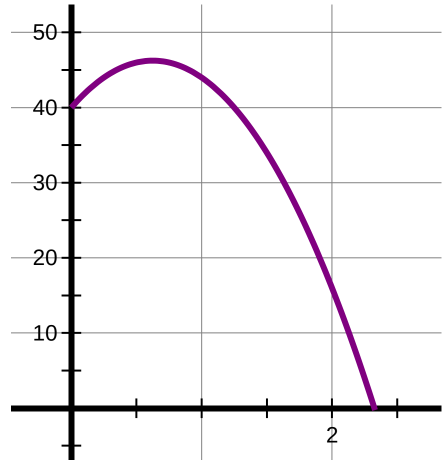
Draw a graph representing total money spent on Starbucks over time. Label the parts of your graph carefully.



(8) After Halloween a rotten pumpkin was tossed from the 3rd floor balcony here at school. The graph below represents the height of the pumpkin, in feet, versus the time, in seconds. Use the graph to answer each question.

(a) Identify a time period over which the average rate of change is zero.

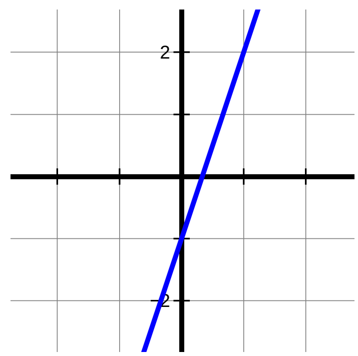
(b) What is the average rate of change from $t = 0$ to $t = 2$?



(9) The graph below represents the equation $ax - 2y = 2$ where a is a constant.

(a) Give a solution to the equation.

(b) Find the value of a. Show all work.



Appendix C

Student Interview Introduction and Questions

Introduction: As you know from class, I am working on a research project for my Master's degree. I am trying to find out more about how students grapple with mathematical representations. I think this is an important area to explore as it shows greater depth of concept understanding and has been shown to help students succeed in future math courses.

I will just ask you some general questions and some questions about your work in class. My intent is to really just understand more about how you understand what we have learned. You are not graded on your responses as I really just need your honest feedback. With the information you provide, I can learn more about how to teach you more effectively!

General History

1. How have you performed in math over the course of high school?

2. What did it mean to show understanding in your previous math classes? Do you feel this class is the same or different? Explain.

3. How have groups or group work been used in your previous math classes?

Class-Specific Questions

4. How did working in groups impact your understanding in the past two units on functions in my class?

5. We completed the following group tasks in class (handouts laid out in front of student). Did any of these help solidify understanding for you? What? How?

6. How is a graph connected to an equation?

7. What are the strengths and weaknesses of representing problem situations using only a graph? Using only an equation?

Conclusion

8. Is there anything we haven't discussed that you would like to add regarding your experience with the groupwork and our functions units?

Appendix D

Group Training Tasks

GROUP BUILDING ACTIVITY 1: SPACESHIP[3]

Objective: select seven persons to go into a spaceship for a voyage to a new planet. You have just been alerted that a giant meteor is on a collision course with earth. The spaceship is equipped to set up life on a new planet. However, you need to select the appropriate people to go aboard. Initially there was room for 12 people; now there is only room for 7. Your group must decide which 7 persons will go to start life on the new planet.

Rules:

1. The group must reach a consensus for each decision.
2. The group members must provide a clear rationale for their decision.

People:

1. A 30 year old male symphony orchestra violin player
2. A 47 year old male priest
3. A 23 year old engineer and her 21 year old husband (they refuse to be separated)
4. A 40 year old policeman who refuses to leave his gun behind
5. An EC senior lineman
6. A 35 year old male high school dropout, recently arrested for armed robbery
7. A 32 year old male high school teacher
8. A 40 year old female doctor (medical)
9. A 50 year old female artist and sculptor
10. A 25 year old male poet
11. A 1 year old female child

GROUP BUILDING ACTIVITY 2: PUZZLED CARDS[3]

Each group will be given an envelope containing different pieces of playing cards. DO NOT open the envelope until I tell you to do so.

Objective: put the pieces together in such a way that each member of your group completes one rectangle.

Rules:
1. Place the pieces carefully in the center of the group.
2. Each group member needs to take at least four pieces. If there are extra pieces left, leave them in the middle of the group.
3. TOTAL silence – no talking, grunting, or sounds of any sort
4. You may not point or signal to other players with your hands in any way.
5. EACH player must put together his or her own puzzle. No one else may show a player how to do it or do it for him or her.
6. This is an exercise in GIVING – you may NOT take a piece from another player, but you may give pieces, one at a time, to any other members of your group by placing the piece next to the other person's pieces.

GROUP BUILDING ACTIVITY 3: MASTER DESIGNER[3]

Each of you will be given a Ziploc bag with 7 pattern pieces inside.

Objective: *individually* create the same design as the Master Designer through the Designer's verbal explanations.

Rules:
1. Set up the cardboard to guard your pieces so that NO other group members can see them.
2. One person, the Master Designer, will create a hidden design using ALL pattern pieces.
3. The Master Designer instructs the other players as to how to replicate his or her design.
4. Players cannot see what the others are doing, nor can they see the design of the Master.
5. Group members may ask questions of the Master Designer.
6. This is an exercise in explaining, questioning, and helping without doing the work for another person!

GROUP BUILDING ACTIVITY 4:
I WANT YOUR DIGITS[4]

Each group will be given one response card and a scratch piece of paper.

Objective: *as a group*, determine the 3 digit number of the other group member in five or fewer guesses.

Rules:

1. One group member sets up cardboard so the other group members cannot see his or her digits; this person writes down a 3 digit number.
2. The other players work TOGETHER to decide on a guess for the 3 digit number explaining their rationale to one another as they present a guess.
3. The group member with the digits tells how many numbers and how many places are correct; the other group members record the numbers and places on the response card.
4. Play continues until the team has obtained the digits!
5. If three digits is too easy, try four digits.

Appendix E

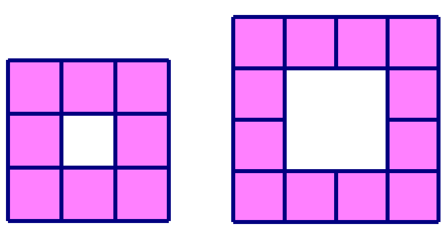
Group-Worthy Tasks

PRECALCULUS
1.4B ACTIVITY[5]
Connecting Representations

SQUARE DONUTS

Examine the pattern below. Work together to complete each of the following tasks. You should use this paper to record notes and ideas; however, you will work as a group to create a poster with all of the information described below.

1. Draw the next two "donuts" following the pattern
2. Create a table relating the donut # and the total shaded squares in the donut
3. Determine an equation relating the donut # to the total shaded squares; write your equation using function notation
4. Create a graph showing the relationship between donut # and total tiles
5. Select one coordinate pair from your table. Identify the same coordinate pair in the figures, the equation, and the graph
6. Explain the relationship between the table, equation, and graph

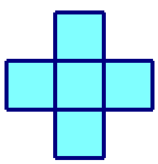


Donut #1 Donut #2

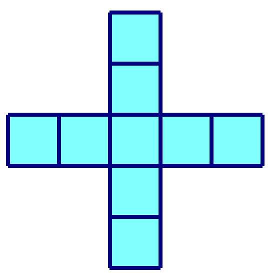
CROSS ARRAYS

Examine the pattern below. Work together to complete each of the following tasks. You should use this paper to record notes and ideas; however, you will work as a group to create a poster with all of the information described below.

1. Draw the next two crosses following the pattern
2. Create a table relating the cross # and the total squares in the cross
3. Determine an equation relating the cross# to the total squares; write your equation using function notation
4. Create a graph showing the relationship between cross# and the total squares
5. Select one coordinate pair from your table. Identify the same coordinate pair in the figures, the equation, and the graph
6. Explain the relationship between the table, equation, and graph



Cross #1



Cross #2

PAINTED CUBES

Each cube below is made up of smaller cubes. Someone has dipped the painted cube in white paint and then decided to create a pattern by painting the smaller cubes that have ONLY two exposed faces in a different color. Examine the pattern. Work together to complete each of the following tasks. You should use this paper to record notes and ideas; however, you will work as a group to create a poster with all of the information described below.

1. Discuss how many smaller cubes will be painted in the different color for the next larger cube.

2. Create a table relating the cube # to the total # of cubes with two faces painted in a different color

3. Determine an equation relating the cube # to the total # of cubes with two faces painted in a different color; write your equation using function notation

4. Create a graph showing the relationship between the cube # to the total # of cubes with two faces painted in a different color

5. Select one coordinate pair from your table. Identify the same coordinate pair in the figures, the equation, and the graph

6. Explain the relationship between the table, equation, and graph

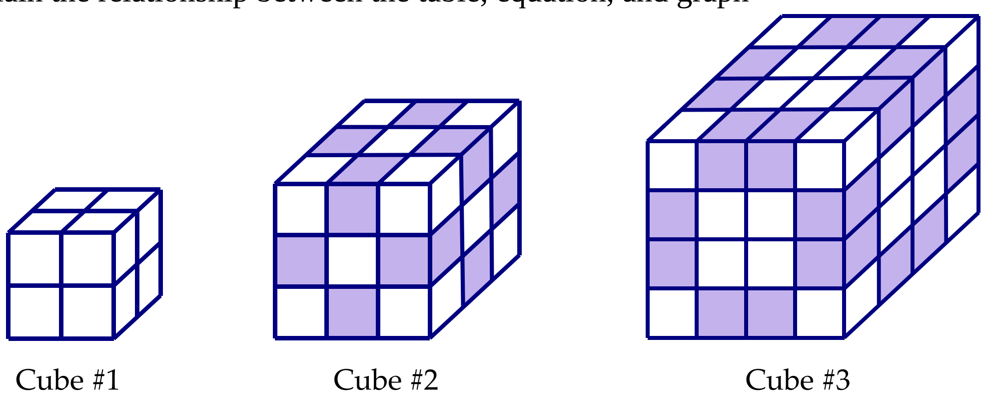
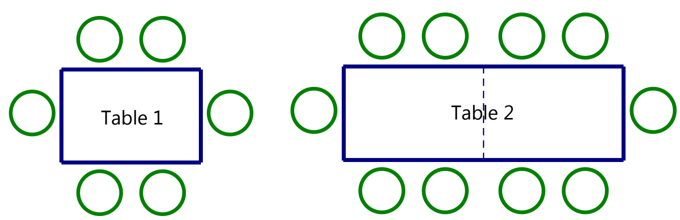


TABLE FOR ?

Each rectangle below represents a restaurant table. The circle on the sides of the tables represent seats. Examine the pattern created when the tables are put together for additional seating. Work together to complete each of the following tasks. You should use this paper to record notes and ideas; however, you will work as a group to create a poster with all of the information described below.

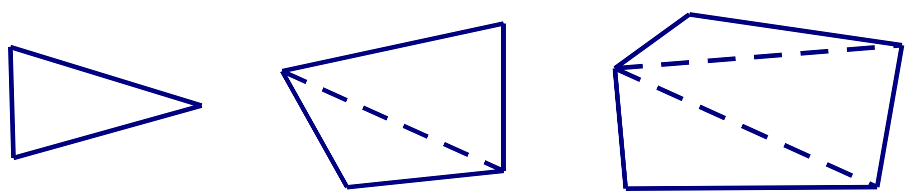
1. Draw the next two table arrangements by following the pattern
2. Create a table of values relating the table # and the total seats
3. Determine an equation relating the table # and the total seats; write your equation using function notation
4. Create a graph showing the relationship between table # and the total seats
5. Select one coordinate pair from your table of values in #2. Identify the same coordinate pair in the figures, the equation, and the graph
6. Explain the relationship between the table, equation, and graph



DIAGONALS IN A POLYGON

The maximum number of diagonals from one vertex are drawn in each polygon below. Examine the pattern as the number of sides in the polygon increases. Work together to complete each of the following tasks. You should use this paper to record notes and ideas; however, you will work as a group to create a poster with all of the information described below.

1. Draw the next two polygons with all possible diagonals from one vertex
2. Create a table related the sides in the polygon to the total diagonals drawn
3. Determine an equation relating the sides in the polygon to the total diagonals drawn; write your equation using function notation
4. Create a graph showing the relationship between sides in the polygon and the total diagonals
5. Select one coordinate pair from your table. Identify the same coordinate pair in the figures, the equation, and the graph
6. Explain the relationship between the table, equation, and graph



PRECALCULUS

Ch. 1A Summary Activity[6]

Representations of Functions

Many high school students work during the summer and put some of their money into savings. This is very helpful when you go away to college and need some spending money. The easiest way to save money is to put a little bit away each week. Four different students - Brittni, Steven, Kyler, and Erik - have different savings plans. Review each plan below and then answer the questions that follow.

BRITTNI

Brittni had some money in savings from last summer. This past summer she put away a little bit of money each week. The table below shows how much Brittni had in her savings account after a given number of weeks.

week	1	2	3	4	5	6
Deposit Amount	250	290	330	370	410	450

ERIK

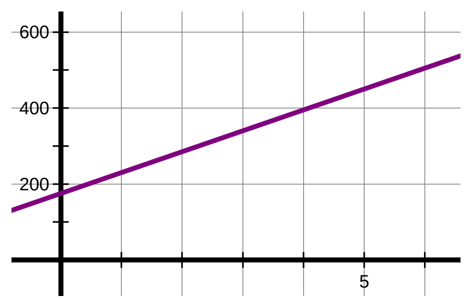
Erik's job automatically deposits a designated amount into his account every week but he can't remember how much. If the balance on his account is $B(x)$ where x is in weeks, Erik does know on two different occasions what the balance was: $B(4) = 550$ and $B(7) = 730$.

KYLER

Kyler's balance, $B(x)$, in his savings account can be represented $B(x) = 625 - 18x$ where x stands for the number of weeks.

STEVEN

The graph shows the balance in Steven's account based on the number of weeks.



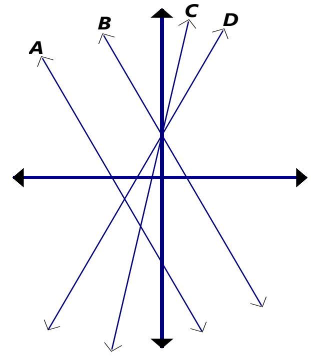
QUESTIONS

1. Which person do you think has the best savings plan? Explain.
2. How does each savings plan compare?
3. Who will have the most money in their savings account at the end of an 8 week summer? Explain how you know.

PRECALCULUS
GROUP PRACTICE TEST[7]
Chapter 1A

(1) Examine the graph and functions below.

(a) Match each function to its appropriate graph.



$f(x) = -2x + 1$
$g(x) = -2x - 2$
$h(x) = 2x + 1$
$p(x) = 5x + 1$

(b) Select <u>one</u> of your answers from above and <u>explain</u> how you determined which graph would match the function.

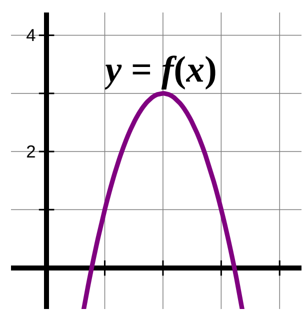
(2) Give an example of a table of values with 5 values that represent a linear function. <u>Explain</u> how you know it is a linear function <u>and</u> give the formula for the function.

(3) Use the graph at right to answer each question.

(a) mark two coordinates on the graph

(b) rewrite the coordinates using function notation

(c) find the rate of change between the two points



(4) True or False? If a line has the equation $3x + 2y = 7$, then the slope is 3. <u>Explain</u> your answer.

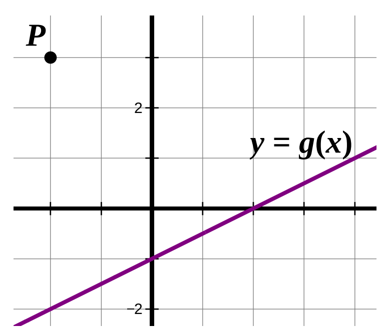
(5) Draw an example of a scatter plot that would have a correlation coefficient of $r = -0.5$.

(6) A linear function $g(x)$ is graphed below. The point P is marked on the graph.

(a) Determine the function $g(x)$.

(b) Find a new linear function that is parallel to $g(x)$ and passes through P.

(c) Find a new linear function that is perpendicular to $g(x)$ and passes through P.



(7) You need to rent a car for your senior trip to Cabo. Rent-a-Dent charges \$70 per day with no mileage charge. Hertz charges \$35 per day and 10 cents per mile. Enterprise charges \$20 per day and 20 cents per mile.

(a) Write a function for the one day charge of <u>each</u> company in terms of x, the miles driven in a day.

(b) What do the starting value and rate of change in each function represent in the context of the problem?

(8) A table for the linear function $p(x)$ is represented below.

x	2	3	4	5	6
$p(x)$	-3	-5	-7	-9	-11

(a) Find a formula for $p(x)$.

(b) Find a formula for the inverse, $p^{-1}(x)$.

(c) Is there more than one way to find $p^{-1}(x)$? <u>Explain</u>.

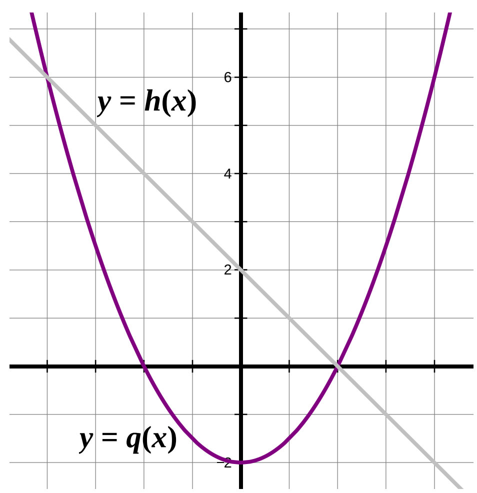
(9) Examine the graph at right to answer each question that follows.

(a) What is $h(0)$?

(b) When does $q(x) = 0$?

(c) When does $h(x) = q(x)$?

(d) What is $q\big(h(-2)\big)$?



PRECALCULUS

1.10 ACTIVITY[8]

Function Machine

Sometimes we are given input and output values and use these values to extrapolate a function. Then, we can use the function to predict additional values.

The Harmonizer in your group has a program called "Function Machine." You will work together to answer each of the questions that follow.

RUN OPTION 1

1. Select option 1 from the program.
2. What is the output goal? What does this mean?
3. Test additional inputs and list the outputs you obtain in the table below.

x						
$f(x)$						

4. What is the equation for the function $f(x)$?
5. What value will give you the output goal?
6. Select an input and output value from above. Describe the input and corresponding output in three different ways.

RUN OPTION 2

1. Select option 2 from the program.
2. What is the output goal? What does this mean?
3. Test additional inputs and list the outputs you obtain in the table below.

x						
$f(x)$						

4. What is the equation for the function $f(x)$?
5. What value will give you the output goal?
6. Select an input and output value from above. Describe the input and corresponding output in three different ways.

RUN OPTION 3

1. Select option 3 from the program.
2. What is the output goal? What does this mean?
3. Test additional inputs and list the outputs you obtain in the table below.

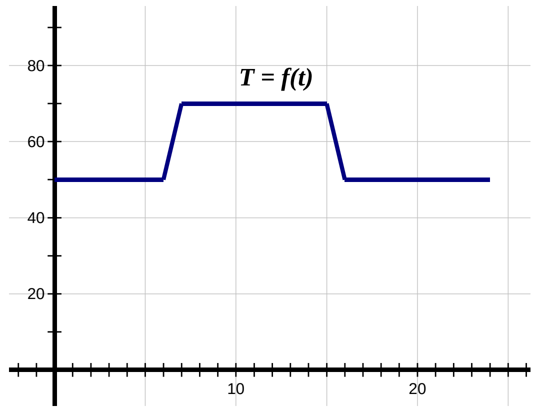
x						
$f(x)$						

4. What is the equation for the function $f(x)$?
5. What value will give you the output goal?
6. Select an input and output value from above. Describe the input and corresponding output in three different ways.

PRECALCULUS
1.12 ACTIVITY[9]
Transforming Functions

GRAPH ONE

1. To save money, the main building at Eastside is kept warm only during school hours. The graph below shows the temperature of the building as a function of time (hours after midnight).



2. What does $f(7) = 70$ mean in the context of this problem?

3. Select four values in the domain and create a table of values below.

t	$f(t)$

4. Many students complain daily that the school is too cold. Because of this, facilities decided to increase the temperature by 5 degrees at each time. Create a table of values using the same times as #3 for the school's new temperature.

t	$q(t)$

5. Graph $q(t)$ on the graph above. How would you write $q(t)$ in terms of $f(t)$?

6. How would the graph change if there was a 2 hour delay for snow? Graph the function, $p(t)$, for a 2 hour delay on the graph above.

7. (a) Select a point from the original temperature graph and write it in function notation.

   (b) Select the shifted point from the 2 hour delay for snow, with the <u>same</u> temperature, and write it in function notation below.

   (c) How could you write $p(t)$ in terms of $f(t)$?

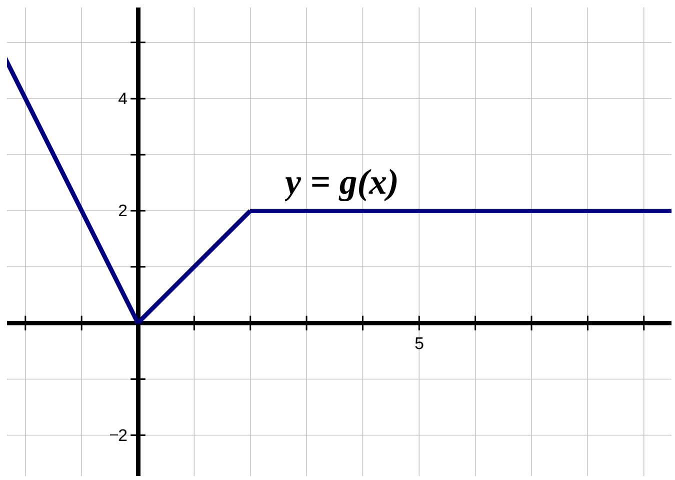
GRAPH TWO

A graph of *g(x)* is shown below.

(a) Suppose $p(x) = g(x) - 2$. How would the graph change?

(b) Suppose $b(x) = g(x + 2)$. How would the graph change?

(c) Select one of the transformations from above and graph it on the same graph.



PRECALCULUS
1.14 ACTIVITY[7]
More with Transformations

Maletta's brother, Dave, is the head graphic designer of a company in San Francisco. TRUE STORY… to create different animations in web design he will oftentimes call me to talk about how to use mathematical functions to define positions for objects.

There are numerous animated remakes of Antoine Dodson's news interview all over the internet. Thus, we will be looking at various transformations of Antoine Dodson's image to see what function transformation will result in the desired image.

IMAGE ONE

Consider the original image of Antoine as $f(x)$. If $T(x)$ is the transformation, how can we write $T(x)$ in terms of $f(x)$? You must justify your answer!

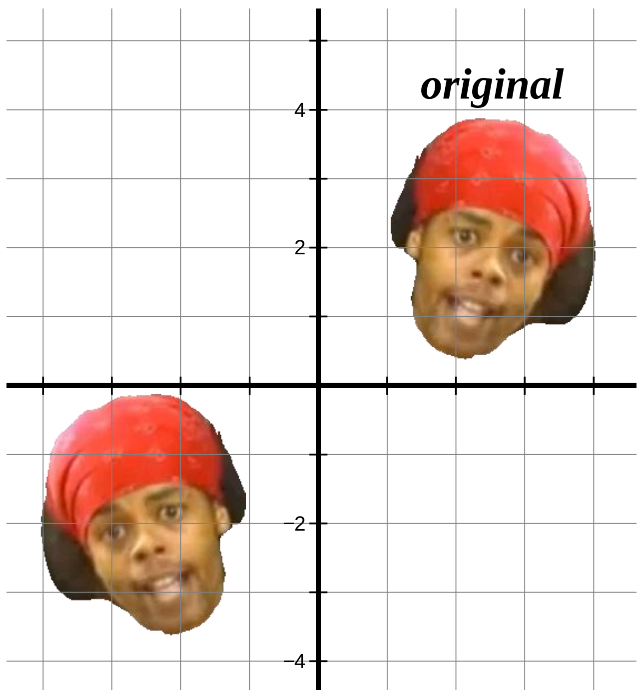


IMAGE TWO

Consider the original image of Antoine as $f(x)$. If $T(x)$ is the transformation, how can we write $T(x)$ in terms of $f(x)$? You must justify your answer!

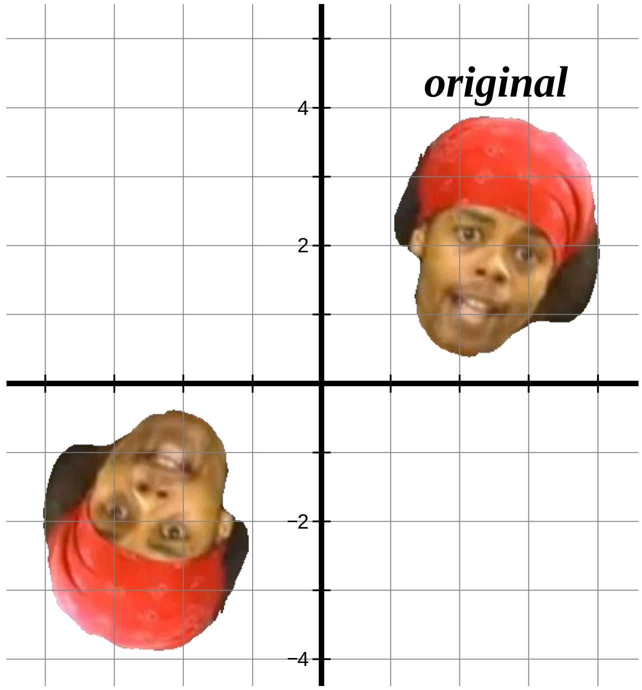


IMAGE THREE

Consider the original image of Antoine as $f(x)$. Determine your own transformation function $T(x)$ that involves a horizontal change and a vertical change. Tell where the final image will end up on the grid and explain how you know.

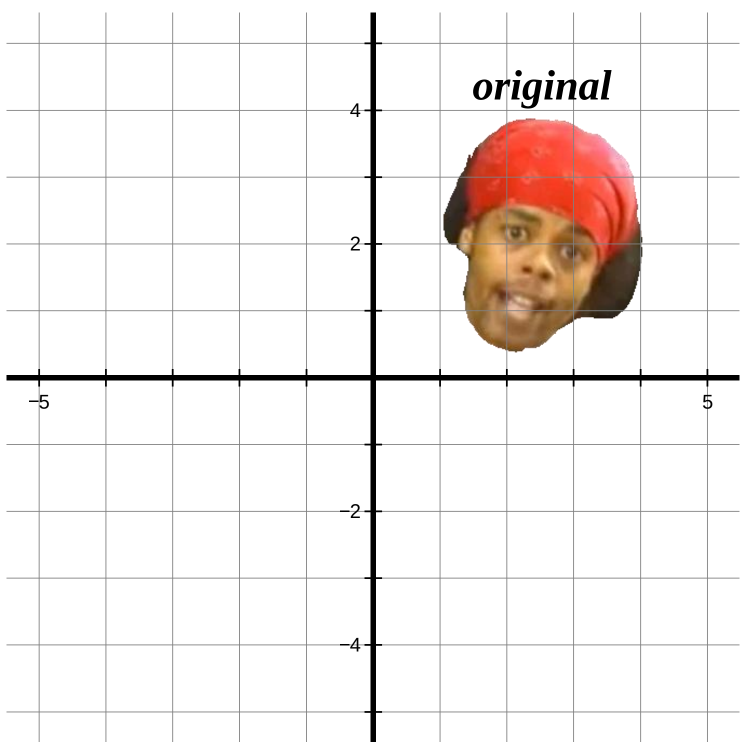
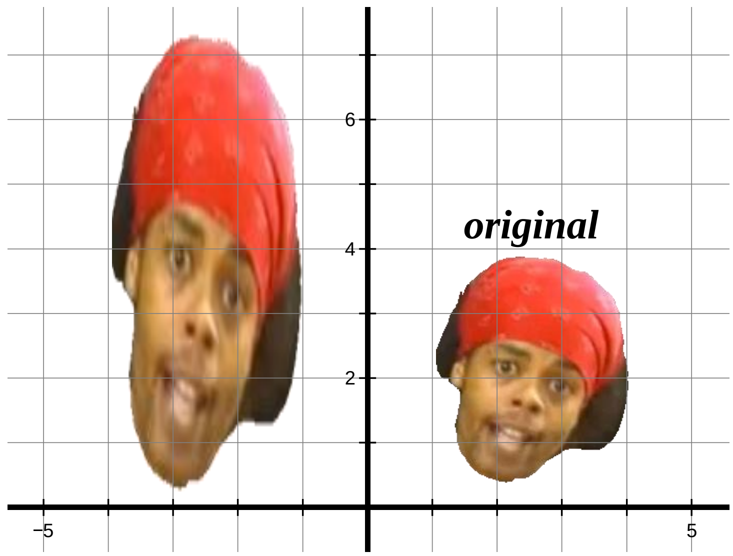


IMAGE FOUR

Consider the original image of Antoine as $f(x)$. If $T(x)$ is the transformation, how can we write $T(x)$ in terms of $f(x)$? You must justify your answer!



PRECALCULUS

1.16 Activity[7]

Function Races

You are going to work together in groups to graph a series of transformations for your given graph – this is a competition between groups and the winners will be handsomely rewarded!

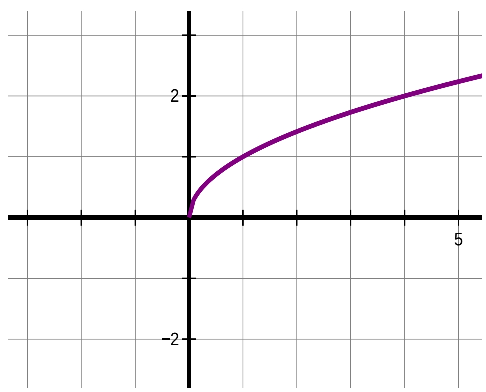
Step 1: Your group gets 30 seconds to record the function transformation on the graph sheet and discuss each step of the transformation.

Step 2: Beginning with the Taskmaster, each person in the group performs ONE transformation. R^2 follows the Taskmaster, then the Harmonizer, and finally the Checker reviews the graph to make sure it is correct.

Step 3: R^2 raises their hand (does NOT yell at me) when the group is finished to see if your group has the correct graph first.

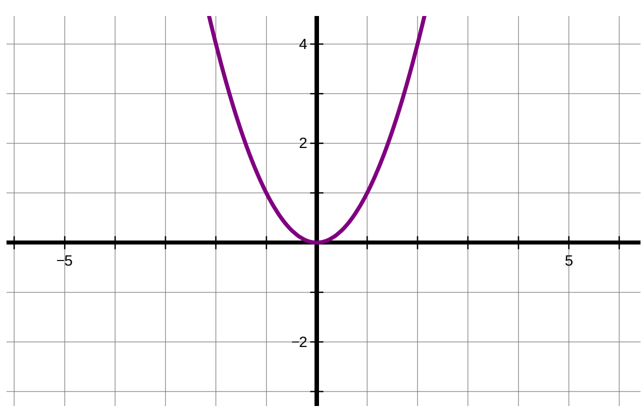
Example:

$$y = -f(x+3) - 2$$



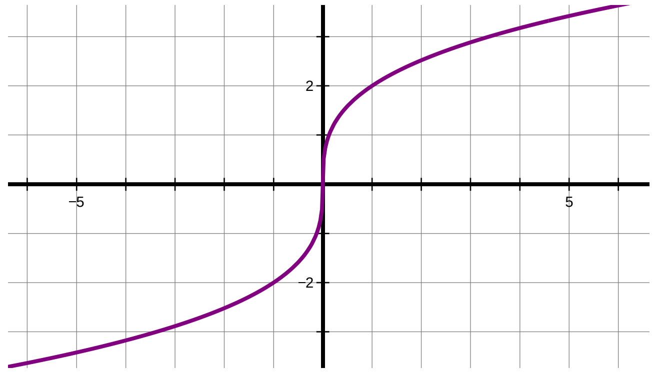
Graph One

$$y = -3f(x+4) + 3$$



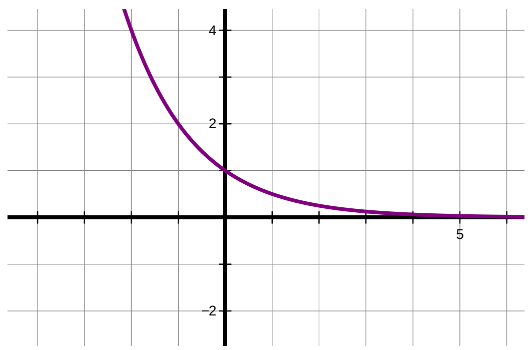
Graph Two

$$y = \frac{1}{2}f(-x) - 1$$



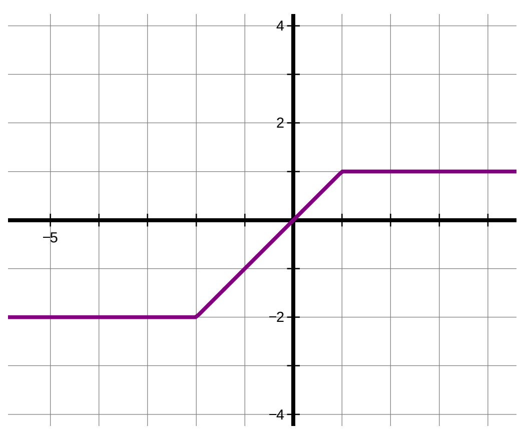
Graph Three

$$y = 2f(x - 3) + 1$$



Graph Four

$$y = \frac{3}{2}f(-x - 3) + 2$$



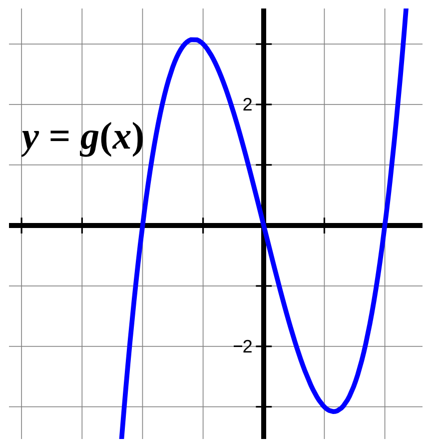
PRECALCULUS
GROUP PRACTICE TEST[7]
Chapter 1B

(1) Let $f(x) = 2x^2 - x$. Evaluate and simplify.
(a) $f(5)$ (b) $f(a)$ (c) $f(a+1)$

(2) Using the graph, complete the table. Then, evaluate each expression.

(a)

x	-2	-1	0	1	2
$g(x)$					



(b) What is $g(-2) + g(0)$?

(c) What is $-2g(1)$?

(d) What is $g(g(-2))$?

(e) When does $g(x) = 0$?

(3) Create your own story that would have a domain of $[0,10]$ and a range of $[1,6]$.

(4) The function $f(x)$ has a range of $(-\infty, 5]$.
(a) Create a graph with the given range.

(b) What is the range of $f(x-3)+4$? Justify your answer using your response to part (a).

(5) During a hurricane, a brick breaks loose from the top of a chimney, 64 feet above the ground. As the brick falls its distance from the ground after t seconds is given by:
$$d(t) = -16t^2 + 64$$

(a) Evaluate $d(1)$. What does $d(1)$ mean in the context of this problem?

(b) How would the transformation $d(t) - 15$ change the problem description?

(c) Suppose in the original function $d(t)$, the brick broke loose at noon, how would the transformation $d(t-2)$ change the situation described?

(d) When does the brick hit the ground in the original function?

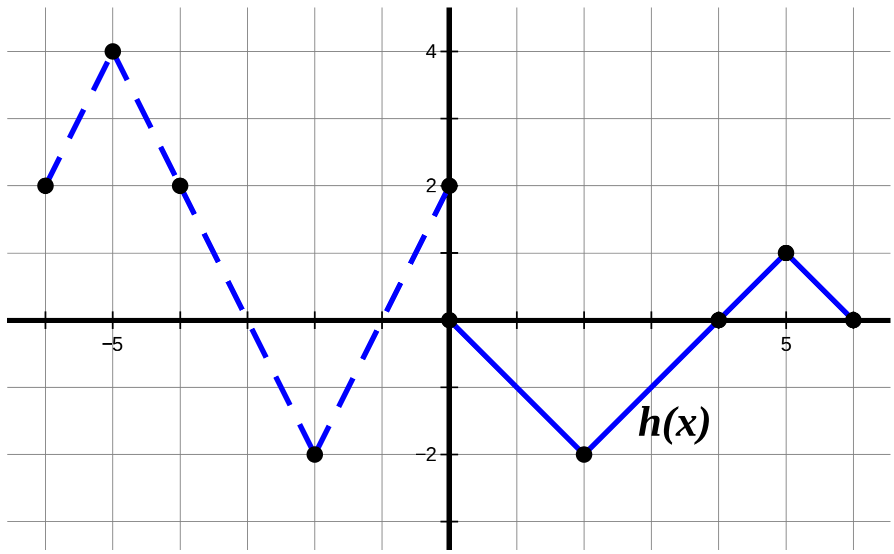
(6) The point $(-2,8)$ is a solution to $f(x)$. What point must be a solution to…

(a) $f(x-3)+11$

(b) $-\frac{3}{4}f(x+2)$

(c) $f(-x+6)$

(7) Find a formula for the transformation of $h(x)$ below. Then, use a point on $h(x)$ to justify your transformation formula!



(8) The table below gives values on the function $g(x)=2f(x-2)-3$. Use the values to determine a table for $f(x)$. You can check your answers to make sure you are correct!

x	-1	0	1	2
$g(x)$	5	-7	3	-11

x				
$f(x)$				

Appendix F

Individual Follow-Up Tasks

PRECALCULUS
CHAPTER 1A: Functions
Check-Up #2[7]

1. Use the table below to answer each question.

x	0	1	2	3	4
$f(x)$	5	2	-1	-4	-7

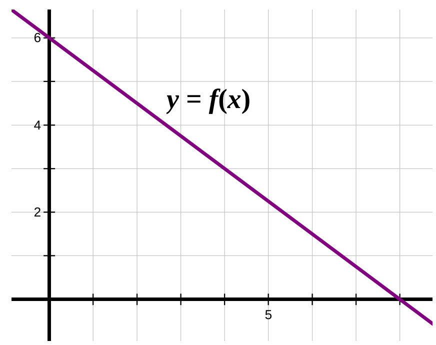
   (a) Is the data in the table linear? Explain how you know.

   (b) What function would model the data in the table?

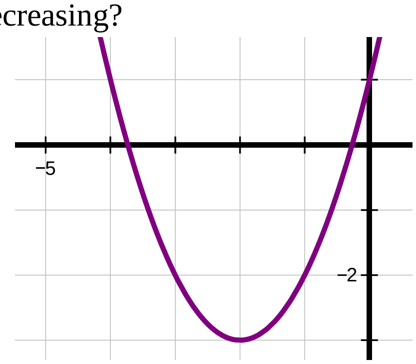
2. Given the graph at right, find each of the following.
   (a) What is $f(4)$?

   (b) When does $f(x) = 0$?

   (c) Describe a situation that would match the graph.



3. For what values of x is the function below increasing? Decreasing?



4. Find a formula for the linear function $f(x)$ given $f(0) = 3$ and $f(4) = -1$.

CHECK UP SELF EVALUATION

<table>
<tr><th></th><th>Unit Learning Targets</th><th>Correct</th><th>Incorrect</th><th>Simple Mistake</th><th>Don’t Get It!</th></tr>
<tr><td rowspan="3">Question 1</td><td>◎ identifies linear functions from tables, graphs, and equations
◎ writes a linear function from a table</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Reflection (if incorrect) – what went wrong, how will you improve, what help is needed?</td></tr>
<tr><td colspan="5">What steps will help you do this type of problem correctly next time?</td></tr>
<tr><td rowspan="3">Question 2</td><td>◎ identifies function values from graphs
◎ writes a situation to match a linear function</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Reflection (if incorrect) – what went wrong, how will you improve, what help is needed?</td></tr>
<tr><td colspan="5">What steps will help you do this type of problem correctly next time?</td></tr>
<tr><td rowspan="3">Question 3</td><td>◎ compares increasing and decreasing intervals</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Reflection (if incorrect) – what went wrong, how will you improve, what help is needed?</td></tr>
<tr><td colspan="5">What steps will help you do this type of problem correctly next time?</td></tr>
<tr><td rowspan="3">Question 4</td><td>◎ writes a linear function from given function values</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Reflection (if incorrect) – what went wrong, how will you improve, what help is needed?</td></tr>
<tr><td colspan="5">What steps will help you do this type of problem correctly next time?</td></tr>
</table>

PRECALCULUS

Follow-Up Problems

Functions

(1) Find the linear function $h(x)$ given $h(0) = 5$ and $h(-2) = 7$.

(2) Find the linear function, $g(x)$ that describes the table below.

x	0	1	2	3	4
$g(x)$	-2	-4	-6	-8	-10

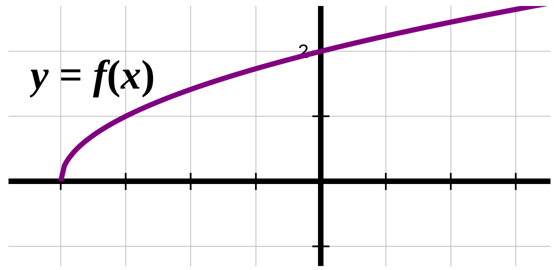
PRECALCULUS

CHAPTER 1B: Function Transformations

Check-Up #1[7]

(1) Use the graph to answer each question that follows.

(a) Complete the table:



(b) What is $f(0) + f(-3)$?

(c) What is $2f(-4)$?

(d) When does $f(x) = 0$?

(2) Let $g(x) = |x - 3| - 2$

(a) What is $g(1)$?

(b) What is $g(a + 2)$?

x	-4	-3	0
$f(x)$			

(c) When does $g(x) = 0$?

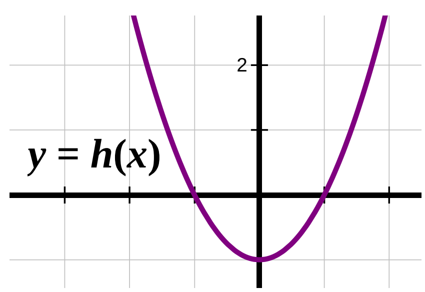
(3) The total application fees that seniors pay is a function of the number of applications they complete. Seniors at Eastside complete between 1 and 10 college applications. Each college application averages $50.

(a) What is the input? What is the output?

(b) Give the domain using appropriate notation.

(c) Give the range using appropriate notation.

(4) The graph of $h(x) = x^2 - 1$ is shown below. Identify a solution for $h(x)$ AND write it three different ways.



CHECK UP SELF EVALUATION

<table>
<tr><th></th><th>Unit Learning Targets</th><th>Correct</th><th>Incorrect</th><th>Simple Mistake</th><th>Don’t Get It!</th></tr>
<tr><td rowspan="3">Question 1</td><td>◎ identifies input and output values from a graph</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Reflection (if incorrect)</td></tr>
<tr><td colspan="5">When finding input and output values from a graph I need to…</td></tr>
<tr><td rowspan="3">Question 2</td><td>◎ evaluates functions for a given value or variable</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Reflection (if incorrect)</td></tr>
<tr><td colspan="5">When evaluating a function I need to…</td></tr>
<tr><td rowspan="3">Question 3</td><td>◎ finds domain and range from a description</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Reflection (if incorrect)</td></tr>
<tr><td colspan="5">When finding the domain and range from a description I need to…</td></tr>
<tr><td rowspan="3">Question 4</td><td>◎ identifies solutions to functions using graphs</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Reflection (if incorrect)</td></tr>
<tr><td colspan="5">To find a solution for a function from a graph I need to…</td></tr>
</table>

Select one of the learning targets from above and explain it in your own words:

____

____

PRECALCULUS
CHAPTER 1B TEST[7]
Function Transformations

/60	%	Grade

Learning Target: finds input and output values from a function, table, and/or graph			
❒ evaluates functions for a number ❒ evaluates functions for a variable expression ❒ finds input values from a given output ❒ completes a table using a graph ❒ evaluates functions using a table and graph ❒ evaluates composite functions using a table and graph ❒ simplifies expressions and/or solves equations accurately			
Exceeds Expectations	**Meets Expectations**	**Approaching Expectations**	**Needs Much Improvement**
All guidelines met (20 points)	5 guidelines met (16 points)	4 guidelines met (12 points)	3 guidelines met (8 points)

(1) Evaluate and simplify given $f(x) = (x+1)^2$

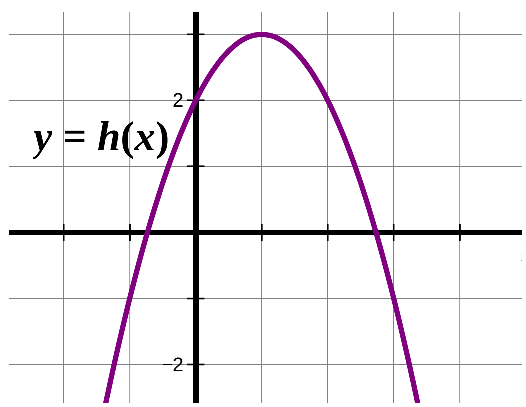
(a) $f(-3)$ (b) $f(a)$ (c) $f(2+a)$

(2) If $g(x) = x^2 - 10$, when does $g(x) = 6$?

(3) Use the graph of $h(x)$ below to complete the table. Then, evaluate each expression.

(a)

x	-1	0	1	2
$h(x)$				



(b) What is $h(0) + h(2)$?

(c) What is $-2h(1)$?

(d) What is $h(h(0))$?

Learning Target: finds domain and range from a graph, table, or description

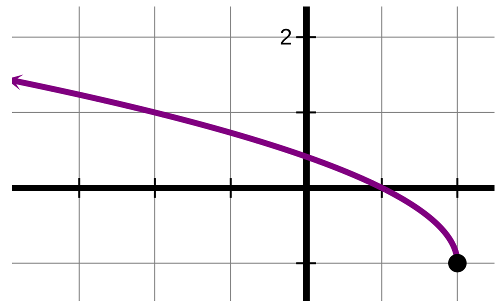
❒ differentiates between domain and range
❒ determines correct domain and range from a graph
❒ identifies and describes function values corresponding to a problem description
❒ determines correct domain and range from a description
❒ uses accurate interval notation when stating domain and range

Exceeds Expectations	Meets Expectations	Approaching Expectations	Needs Much Improvement
All guidelines met (10 points)	4 guidelines met (8 points)	3 guidelines met (6 points)	2 guidelines met (4 points)

(4) A graph of $f(x)$ is shown at right.

(a) What is the domain? Make sure to use correct notation!

(b) What is the range? Make sure to use correct notation!



(5) 80 meters of fencing can be used to enclose a rectangular area of anywhere from 39 square meters to 400 square meters. The area, A is a function of the length of **ONE** side of the rectangle, x.

(a) The smallest area can be represented as $A(39) = 39$. Explain what this means in the problem context.

(b) The largest area can be represented as $A(20) = 400$. Explain what this means in the problem context.

(c) What are all possible values of the domain? Make sure to use correct notation!

(d) What are all possible values of the range? Make sure to use correct notation!

Learning Target: identifies vertical and horizontal shifts of functions

❒ describes horizontal and vertical transformations
❒ finds correct slope and intercept for a linear function from a table
❒ determines a table for a horizontal and vertical transformation of a function
❒ graphs a horizontal transformation
❒ graphs a vertical transformation
❒ interprets a written description to write a horizontal or vertical transformation of a function

Exceeds Expectations	Meets Expectations	Approaching Expectations	Needs Much Improvement
All guidelines met (20 points)	5 guidelines met (16 points)	4 guidelines met (12 points)	3 guidelines met (8 points)

(6) Describe how the graph of $f(x)$ transforms if $y = f(x + 2) - 3$.

(7) The table below represents a linear function $g(x)$.

x	-2	-1	0	1	2
$g(x)$	12	8	4	0	-4

(a) What is the formula for $g(x)$?

(b) If $p(x) = g(x-5) + 1$, complete the table for $p(x)$ below.

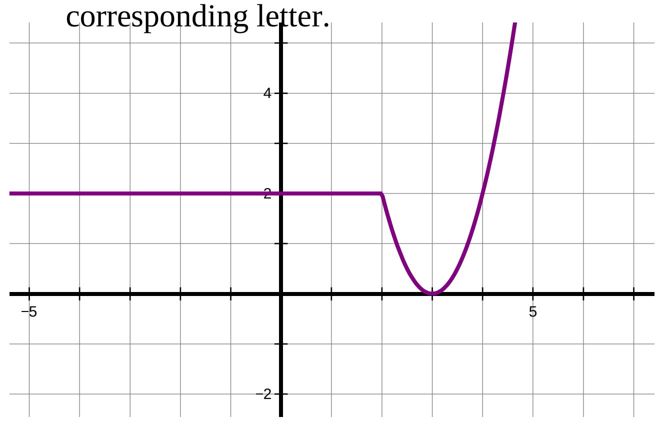
x					
$p(x)$					

(c) What is the formula for $p(x)$?

(8) Use the graph of $f(x)$ below to graph each transformation. Please label your graph with the corresponding letter.

(a) $f(x-2)$

(b) $f(x) + 3$



(9) Each day Karthik takes a taxi to work. The trip is x miles and the cost of the trip is $f(x)$. Write a transformation of $f(x)$ for each description.

(a) Karthik decided to add a tip of $5 onto the cost of the taxi ride.

(b) Karthik had a new driver one day that got lost and had to pay for 2 extra miles.

Learning Target: reflecting functions			
❒ identifies transformations from a graph ❒ writes transformations of functions using appropriate notation ❒ proves transformation formula using a point ❒ written explanation is accurate and includes appropriate mathematical vocabulary			
Exceeds Expectations	**Meets Expectations**	**Approaching Expectations**	**Needs Much Improvement**
All guidelines met (5 points)	3 guidelines met (4 points)	2 guidelines met (3 points)	1 guideline met (2 points)

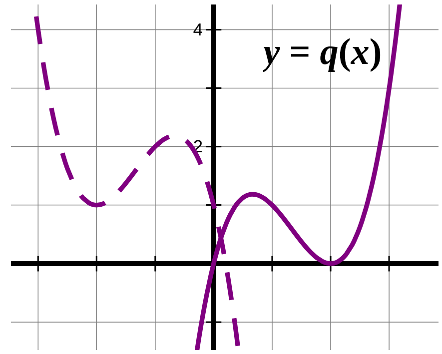
(10) Explain the difference in transformation between $y = -f(x)$ and $y = f(-x)$.

(11) Examine each graph below. The graph shown in dashed lines is a transformation of $q(x)$.

(a) Describe the transformation of $q(x)$.

(b) Write the transformation in terms of $q(x)$.

(c) Prove your transformation above is correct by selecting a point on $q(x)$ then showing how the transformations result in the corresponding point on the dashed graph.



Learning Target: identifies and writes multiple transformations of a function			
❐ writes reflections of points ❐ writes vertical stretches of points ❐ writes horizontal and vertical transformations of points ❐ writes a point with a horizontal reflection and shift			
Exceeds Expectations	**Meets Expectations**	**Approaching Expectations**	**Needs Much Improvement**
All guidelines met (5 points)	3 guidelines met (4 points)	2 guidelines met (3 points)	1 guideline met (2 points)

(12) The point $(3, -2)$ is a solution to $f(x)$. What point will be a solution to each transformation of $f(x)$ below?

(a) $2f(-x) - 3$ (b) $-\frac{1}{2}f(x+2)$ (c) $f(-x-5) + 4$

Footnotes

[1]The space provided for student responses has been removed to save paper; Question 3(b) was removed from the scoring as described earlier. Some questions were adapted from the following sources: Connally (2006), Even (1998), Knuth (2000a & 2000b), and Rider (2007).

[2]The space provided for student responses has been removed to save paper. Some questions were adapted from the following sources: Connally (2006), Even (1998), Knuth (2000a & 2000b), and Rider (2007).

[3]Adapted from Cohen (1994).

[4]Adapted from the College Preparatory Mathematics geometry text (2007).

[5]Some of these problems were adapted from the Discovering geometry text (1997) and the NCTM special focus issue – Developing mathematical understanding through mathematical representations for Mathematics teaching in the middle school (2008) as well as Driscoll (1999). Each group was given a different pattern task.

[6]This task was adapted from Coulombe & Berenson (2001). The space provided for student responses has been removed to save paper.

[7]The space provided for student responses has been removed to save paper.

[8]This activity comes from Texas Instruments (TI) and uses a calculator program. I used the program from the TI activity (http://education.ti.com/calculators/downloads/US/Activities /Detail?ID=11572) and created my own questions. The space provided for student responses has been removed to save paper.

[9]These problems were adapted from Connally's Functions modeling change text (2007). The space provided for student responses has been removed to save paper.