1. Find, in the form $y = f(x)$, the general solution to the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + xy = x + 1$$

Standard form is $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x} y = \frac{x+1}{x^2}$ and integrating factor is $v = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$

Thus the differential equation becomes $x \frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{x+1}{x}$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(xy) = 1 + \frac{1}{x} \Rightarrow xy = \int 1 + \frac{1}{x}\,\mathrm{d}x = x + \ln x + c \Rightarrow y = 1 + \frac{\ln x + c}{x}$$

2. Find the solution to the differential equation

$$\sqrt{x^2+1}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2y}\ ,$$

Separate variables

$$2y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sqrt{x^2+1}} \Rightarrow \int 2y\,\mathrm{d}y = \int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x \Rightarrow y^2 = \sqrt{x^2+1} + c \Rightarrow y = \sqrt{\sqrt{x^2+1}+c}$$

3. Solve the differential equation $x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cos x$, given that the $y = \frac{2}{\pi}$ when $x = \frac{\pi}{2}$.

Standard form is $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x} y = \frac{\cos x}{x}$ and integrating factor is $v = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$

Thus the differential equation becomes

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x\cos x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^2 y) = x \cos x$$

$$\Rightarrow x^2 y = \int x\cos x\,\mathrm{d}x = x\sin x + \cos x + c \text{ so } y = \frac{x\sin x + \cos x + c}{x^2}$$

Now substituting $y = \frac{2}{\pi}$ when $x = \frac{\pi}{2}$ we solve for c

$$\frac{2}{\pi} = \frac{\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} + c}{(\frac{\pi}{2})^2} = \frac{\frac{\pi}{2} + c}{(\frac{\pi}{2})^2} \Rightarrow \frac{\pi}{2} = \frac{\pi}{2} + c \Rightarrow c = 0$$

so finally

$$y = \frac{x\sin x + \cos x}{x^2}$$

4. Solve the differential equation $x^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} + e^{1/x} = 0$, given that $y = 0$ when $x = -1$.

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = -e^{\frac{1}{x}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2} e^{\frac{1}{x}} \Rightarrow y = \int -\frac{1}{x^2} e^{\frac{1}{x}} \,\mathrm{d}x = e^{\frac{1}{x}} + c$$

when $x = -1$ then $y = 0$ so we solve for c

$$0 = e^{-1} + c \Rightarrow c = -e^{-1} \text{ so } y = e^{\frac{1}{x}} - e^{-1}$$

5. A element A decays to another element B which is also radioactive. A is called the mother nucleus and B is called the daughter nucleus (why radioactive elements are considered feminine is open to question!). Suppose the half life of element A is 30 minutes and the half life of element B is 10 minutes and that you start off with 1.0 moles of A and zero moles of B. Your task in this problem is to find the number of moles of B as a function of time t in minutes. This of course depends on the number of moles of A at time t.

   (a) Show that the number of moles of A at time t is $A = 1.0e^{-k_A t}$ where $k_A = \ln 2/30 \text{ min}^{-1}$.
   Now Now, don't get your hopes up. You'll have to wait until next week – this is for homework.

   (b) The rate of change in the number of moles of B equals the rate of increase in B due to the decay of A minus the rate of decrease in B due to the decay of B. That is

   $$\frac{\mathrm{d}B}{\mathrm{d}t} = k_A A - k_B B$$

   where $k_B = \ln 2/10 \text{ min}^{-1}$. Substitute for A in the above differential equation and solve it for B as a function of time. Sketch the function using your graphical calculator.

   (c) What is the maximum number of moles of B and when is it at a maximum?