1. A series is defined recursively as follows

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

where $a_1 = 1$. Find the limit of a_n as $n \to \infty$.

Assume $\lim_{n\to\infty} a_n = x$ then $\lim_{n\to\infty} a_{n+1} = x$ also. Substituting these results into the above equation yields

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) \Rightarrow 2x = x + \frac{2}{x} \Rightarrow x^2 = 2 \Rightarrow x \pm \sqrt{2}$$

Since $a_n > 0$ for all n then $\lim_{n\to\infty} a_n = \sqrt{2}$

2. Determine whether the following series converge or diverge

(a) $\sum_1^\infty \frac{n^2}{n!}$

Apply the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2} = \frac{(n+1)^2}{(n+1)n^2} = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2}$$

so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n^2}\right) = 0\ .$$

Now, since $0 < 1$ the series converges by the ratio test.

(b) $\sum_1^\infty \frac{1}{n(n+1)}$

Application of the ratio test for this series gives $\lim_{n\to\infty} a_n = 1$ so the ratio test fails. Try the integral test. $a_n = \frac{1}{n(n+1)}$ so test the integral $\int_1^\infty \frac{1}{x(1+x)}\, \mathrm{d}x$ for convergence

$$\int_1^\infty \frac{1}{x(1+x)}\,\mathrm{d}x = \int_1^\infty \left(\frac{1}{x} - \frac{1}{1+x}\right)\mathrm{d}x = [\ln x - \ln(1+x)]_1^\infty = \left[\ln\left(\frac{x}{1+x}\right)\right]_1^\infty = \ln 2\ .$$

The integral converges so the series converges.

3. Find the interval of convergence of the following power series

$$1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \cdots$$

$a_n = 2^n x^n$. For convergence apply the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right| = \lim_{n\to\infty}|2x| = 2|x| < 1$$

So for convergence $|x| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$. Now test the end points. When $x = \frac{1}{2}$,

$$\sum_n^\infty a_n = 1 + 1 + 1 + 1 \cdots$$

so the series diverges. When $x = -\frac{1}{2}$,

$$\sum_n^\infty a_n = 1 - 1 + 1 - 1 + 1 - 1 \cdots$$

so the series diverges also. So the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

4. Find the first three non zero terms of the Maclaurin series for the function $f(x) = e^{-x^2}$

There are two methods. First the long method

$$\begin{aligned}
f(x) &= e^{-x^2} & f(0) = 1 && \Rightarrow c_0 = 1\\
f'(x) &= -2xe^{-x^2} & f'(0) = 0 && \Rightarrow c_1 = 0\\
f''(x) &= -2e^{-x^2} + 4x^2e^{-x^2} & f''(0) = -2 && \Rightarrow c_2 = -\frac{2}{2!} = -1\\
f'''(x) &= 4xe^{-x^2} + 8xe^{-x^2} - 8x^3e^{-x^2}\\
&= 1\ 2xe^{-x^2} - 8x^3e^-x^2 & f'''(0) = 0 && \Rightarrow c_3 = 0\\
f''''(x) &= 12e^{-x^2} - 48x^2e^{-x^2} + 16x^4e^{-x^2} & f''''(0) = 12 && \Rightarrow c_4 = \frac{12}{4!} = \frac{1}{2}
\end{aligned}$$

So the Maclaurin series is $f(x) = 1 - x^2 + \frac{1}{2}x^4 + \cdots$.

The short method makes use of the Maclaurin series for e^x which is

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$$

Then it follows that

$$\begin{aligned}
e^{-x^2} &= 1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \frac{1}{3!}(-x^2)^3 + \cdots\\
&= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{3!}x^6 + \cdots
\end{aligned}$$