1. In an environment with plentiful food and no predators rabbits will grow without bound at a rate which is proportional to the current population. A suitable rate equation for this situation would be.

$$R' = kR$$

where $R$ is the number of rabbits $R'$ is the yearly growth rate and $k$ is a constant which is often refereed to as the rate constant. Suppose the initial rabbit population is 150 and the initial growth rate is 90 rabbits per year.

   (a) What is the value of $k$ and what are its units?
   Putting $R' = 90$ and $R = 150$ we find $90 = 150k$ so $k = 90/150 = 0.6$ per year.

   (b) Use an Euler approximation with a time step of one year to find the number of rabbits after one year. (Calculate this by hand)
   $R_1 = R_0 + R'_0 = 150 + 90 = 240$ rabbits

   (c) Improve your estimate of the population after one year by taking two half year steps. (Calculate this by hand).
   First, after half a year $R_{0.5} = R_0 + R'_0(0.5) = 150 + 45 = 195$ rabbits. Now the new rate is $R'_{0.5} = kR_{0.5} = (0.6)(195) = 117$ rabbits per year. So $R_1 = R_{0.5} + R'_{0.5}(0.5) = 195 + 117(0.5) = 253.5$ rabbits. This estimate is higher.

2. If predators such as foxes are introduced into the rabbit population the situation changes. In this case the rate constant $k$ will depend on the number of foxes, $F$. Suppose we replace $k$ in the above rate equation with $k = a + bF$.

   (a) What should the units for the constants $a$ and $b$ be?
   Since the units for $k$ are "per year" the units for $a$ are "per year" and the units for $b$ are "per year per fox".

   (b) Why would it make sense for the coefficient $b$ to be negative?
   The more foxes there are the more rabbits that would be killed and hence the smaller the growth rate.

3. With foxes and rabbits living together an interesting dynamic occurs. The growth rate of rabbits depends on both the rabbit population $R$ and the fox population $F$. The growth rate of the foxes will also depend on these populations. A suggested model for the yearly growth rate of rabbits and foxes is

$$\begin{aligned} R' &= (a + bF)R \\ F' &= (c + dR)F \end{aligned}$$

Suppose that research shows that the best coefficients for this model are $a = 0.6$, $b = -0.04$, $c = -0.2$ and $d = 0.002$.

(a) With an initial rabbit population of 150 and an initial fox population of 10 find the initial growth rates for the rabbit and fox populations.
The rabbit growth rate is $R' = (0.6 - 0.04(10))150 = 30$ rabbits per year.
The fox growth rate is $F' = (-0.2 + 0.002(150))10 = 1$ fox per year.

(b) Use these rates to find an estimate for the population of rabbits and foxes after one year.
The number of rabbits after one year would be $R_1 = R_0 + R'_0 = 150 + 30 = 180$ rabbits.
The number of foxes after on year would be $F_1 = F_0 + F'_0 = 10 + 1 = 11$ foxes.

(c) Refine your estimate for the population at the end of one year using an Euler approximation with time step equal to half a year. (Calculate this by hand).
First, after half a year there would be $R_{0.5} = R_0 + R'_0(0.5) = 150 + 15 = 165$ rabbits and $F_{0.5} = F_0 + F'_0(0.5) = 10 + 0.5 = 11.5$ foxes.
Now the new rates are
$R'_{0.5} = (0.6 - 0.04(F_{0.5}))R_{0.5} = (0.6 - 0.04(10.5))(165) = 29.7$ rabbits per year and
$F'_{0.5} = (-0.2 + 0.002(R_{0.5}))F_{0.5} = (-0.2 + 0.002(165))(10.5) = 1.365$ foxes per year.
So after one year would be $R_1 = R_{0.5} + R'_{0.5}(0.5) = 165 + 29.7(0.5) = 179.85$ rabbits.
and $F_1 = F_{0.5} + F'_{0.5}(0.5) = 10.5 + 1.365(0.5) = 11.1825$ foxes.

(d) Write a simple program that uses Euler's method to calculate estimates of the populations after one year, accurate to the nearest decimal place. What time step did you need to use? Include your code with your answer. After one year the populations are $R = 178.7$ rabbits and $F = 11.4$ foxes accurate to one decimal place. I needed a time step of $dt = 1/128$ to confirm this.

(e) Now run your model for 20 years using the same time step as above.

(i) What is the maximum and minimum rabbit population during the 20 year period?
The maximum population is 200.8 rabbits and the minimum is 40.4 rabbits. To get this level of accuracy you need a time step of $dt = 1/2^{12}$.

(ii) What is the maximum and minimum fox population during the 20 year period?
The maximum population is 22.9 foxes and the minimum is 9.2 foxes.

(f) When you ran your program for 20 years you should have seen that the populations did not settle on any particular values but oscillated. These oscillations occur for most initial populations. However, there are unique initial populations of rabbits and foxes that are stable – ie that will not change over time. Find these populations.
If the populations do not change then $R' = 0$ and $F' = 0$. Setting the rate equations to zero yields $R = 150$ rabbits and $F = 15$ foxes are the stable populations.