- This test forms part of our assessment of your personal learning in this program. You may not collaborate with other people, but you may consult your text book and notes.
- Attempt all questions on this test. Do not leave answers blank. Marks will be given for partial answers so show all your working.
- Your completed test is due at 9:00 am on Monday Feb 2nd.

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1. The Vanishing Elephants. According to one environmental group, the population of wild elephants declines by 5% per year. If E denotes the population t years from now, then we can express the decline by a rate equation of the form

$$E' = -kE$$

(a) What value would you give k? (Think carefully)

$k = 0.05$ since the elephant population changes by 5% of given population per year.

(b) Suppose the initial number of elephants is E_o. Write down an expression for the number of elephants after one year assuming that the rate is constant for an entire year. Now repeat the procedure for a second year with the rate for the second year. Generalize your result so that you have an expression for the number of elephants after n years.

$E_1 = E_0 + E_0' \Delta t = E_0 - 0.05E_0$ since Δt is taken as one year. So $E_1 = (0.95)E_0$. Similarly $E_2 = (0.95)E_1 = (0.95)^2 E_0$. By induction $E_n = (0.95)^n E_0$

(c) If the initial population is 200,000 how many does your model predict will remain after 10 years. How many years will it take for the population to be cut in half to 100,000?

The number of elephants after 10 years would be $E_10 = 200,000(0.95)^10 = 119,700$
The amount of time it takes for the elephant population to drop to 100,000 can by setting $E_n = 100,000$ and solving for n. Thus $100,000 = 200,000(0.95)^n \Rightarrow (0.95)^n = 0.5 \Rightarrow n = \log(0.5)/log(0.95) = 13.5$. So it takes over 13 years for the elephant population to reduced by half.

(d) Consider this argument: "Since 1/20th of the population disappears each year, in 20 years the population will vanish completely." Does your rate equation predict that the elephant population will vanish in 20 years? What, if anything, is wrong with the argument quoted in the first sentence?

The rate equation doesn't predict this since the statement assumes that a constant amount corresponding to 1/20th of the initial population is reduced each year. Since the population is decreasing each year the number that are removed each year is also decreasing.

(e) In your model above you assumed that the rate was constant over an entire year. Now refine your model to correct this problem by using the rate equation with smaller step sizes. Starting with 200,000 elephants find out how many elephants there will be after 10 years. You should use a computer for this problem and decrease your step size until you have are accurate to the nearest 100 elephants.

See Barry's code on the website for suitable code. The answer is $E_10 = 121,300$.

(f) With your new model how long does it take for the population to reduced by half?

Using your program you can show that it takes 13.9 years to reduced by half.

2. The arms race model you looked at in the in class test was developed by L.F. Richardson in 1920's as a way to explain the arms build up that lead to the first world war. He considered the model

$$\begin{aligned} x' &= -4x + 2y \\ y' &= 5x - 4y + 12 \end{aligned}$$

where x and y are the annual military budgets of the two countries (in billions of dollars)

(a) The following questions explore explore the meaning of the coefficients in these rate equations. To answer the questions it might be helpful to consider situations where either x or y or both are zero.

(i) What are the units of each of the coefficients?

All the coefficients except the constant term "12" have units of "per year" or alternatively "billions of dollars per year per billion dollars". The units of the constant term are "billions of dollars per year.

(ii) Explain why it makes sense for the coefficient of x in the first equation and the coefficient of y in the second equation to be negative?

These coefficients are negative because they reflect the fact that there are pressures for countries to *decrease* their military spending when their own military spending gets high.

(iii) Explain why it makes sense for the coefficient of y in the first equation and the coefficient of x in the second equation should be positive?

These coefficients are positive because they reflect the natural tendency for countries to *increase* their military spending in response to the large military spending of their neighbours.

(iv) What does the additional constant 12 in the second equation tell you about the military spending habits of country Y.

This coefficient reflects that country Y has a natural tendency to increase its spending even when its neighbour's military spending is zero. One might call it a belligerent, bellicose or bushy nation.

(b) In the in class test you used the model to predict the spending in one year based on a spending of $x = 5$ and $y = 6$ this year. You should have arrived at the answer $x = -3$ and $y = 19$. You may have worried about what $x = -3$ means. In fact this is an artifact of the approximation that the rate at which spending changes is constant during the entire year.

(i) Improve your estimate of the budgets in 1 year by taking two half year steps. Do this by hand. Is the estimate any better?

The rate equations give the initial rates as $x'_0 = -4(5) + 2(6) = -8$ and $y'_0 = 5(5) - 4(6) + 12 = 13$. Using these rates we find new values for x and y after half a year using Euler's method:

$$\begin{aligned} x_{0.5} &= 5 - 8(0.5) = 1 \\ y_{0.5} &= 6 + 13(0.5) = 12.5\ . \end{aligned}$$

We now repeat the procedure with these new values to get new rates $x'_{0.5} = -4(1) + 2(12.5) = 21$ and $y'_0 = 5(1) - 4(12.5) + 12 = -33$. So that the values for x and y after one year are

$$\begin{aligned} x_1 &= 1 + 21(0.5) = 11.5 \\ y_1 &= 12.5 - 33(0.5) = -4\ . \end{aligned}$$

This result is not much better since now y has a negative value.

(ii) Improve your estimate again by taking 4 quarter year steps. Again do this by hand. Are your answers approaching a limit?

Repeating the above procedure with $\Delta t = 0.25$ gives $x = 4.39$ and $y = 7.22$. These answers are not sufficiently close to the other approximations above to be sure that they are approaching a limit. A smaller time step is needed.

(iii) To find a good estimate for the budgets in one year you should make much smaller steps. Write a simple program to find the expected budget in one year which is accurate to 2 decimal places.

The code predicts that $x = 3.94$ and $y = 7.91$. See Barry's sample code on line

(c) Now you are ready to explore what the model predicts in 10 years. Using the same step size as in (b)(iii) adjust your model so that you can draw a graph of y vs x for t between 0 and 10 years. What are the final values of x and y after 10 years? Are they much different than they were after 1 year?

After 10 years $x = 4.00$ and $y = 8.00$. These values do not differ much.

(d) Repeat the above procedure for different initial values of x and y. Use

(i) $x = 0$ $y = 0$. After 10 years $x = 4.00$ and $y = 8.00$.

(ii) $x = 8$ $y = 10$. After 10 years $x = 4.00$ and $y = 8.00$.

(iii) $x = 2$ $y = 12$. After 10 years $x = 4.00$ and $y = 8.00$.

What is similar about the behaviour of the models in each of these cases? Explain this behaviour based on the original rate equations.

All the results approach the same values after 10 years. The point (x, y) is a fixed point of the rate equations. The rate equations show that for those values the rate of change of x and y are both zero. The fact that all initial conditions tend to this fixed point show that it is a *stable* fixed point.