

Astrophysics Ce10 - Physics of Astronomy - spring 2013 weeks 3-4
 solutions - Zita

1, 3, 4, 5, 6, 7, 8 11, 13, 14, 15, 16

due Thurs Apr 04

- 10.1 Show that the equation for hydrostatic equilibrium, Eq. (10.7), can also be written in terms of the optical depth τ , as

$$\frac{dP}{d\tau} = \frac{g}{\kappa}$$

This form of the equation is often useful in building model stellar atmospheres.

$$(10.7) \frac{dp}{dr} = - \frac{GM_r \rho}{r^2} = - \rho g, \quad (9.13) \frac{dc}{dr} = - \kappa p$$

$$\frac{dp}{dc} = \frac{dp}{dr} \frac{dr}{dc} =$$

- 10.3 Assuming that 10 eV could be released by every atom in the Sun through chemical reactions, estimate how long the Sun could shine at its current rate through chemical processes alone. For simplicity, assume that the Sun is composed entirely of hydrogen. Is it possible that the Sun's energy is entirely chemical? Why or why not?

$$Power = L_0$$

$$= 3.83 \times 10^{33} \frac{\text{erg}}{\text{s}}$$

$$m_H =$$

$$\text{Mass of Sun} = \text{mass of H} \times \text{number of H}$$

$$M_0 = m_H N \rightarrow N =$$

$$\text{Energy released} = \text{energy per particle} \times \text{number of particles}$$

$$E = \epsilon N =$$

$$\text{Power} = \frac{\text{Energy}}{\text{time}} \rightarrow \text{time} =$$

- 10.4 (a) What temperature would be required for two protons to collide if quantum mechanical tunneling is neglected? Assume that nuclei having velocities ten times the root-mean-square (rms) value for the Maxwell-Boltzmann distribution can overcome the Coulomb barrier. Compare your answer with the estimated central temperature of the Sun. $T_{\text{center}} \sim 10^7 \text{ K}$

$$V = 10 V_{\text{rms}}$$

reduced mass for addition between 2 protons

Thermal energy = Kinetic energy

$$\frac{3}{2} kT = \frac{1}{2} \mu V_{\text{rms}}^2 \quad \text{where } \mu = \frac{m_p m_p}{m_p + m_p} =$$

$$\text{If } V = 10 V_{\text{rms}} =$$

Can over come the Coulomb barrier:

Electric potential energy = Kinetic energy

$$(\text{cgs}) \quad \frac{e^2}{r} = \frac{1}{2} \mu V^2 = \text{OKT}$$

Separation between charges: assume $(P) \uparrow 1 \text{ fm} \downarrow r = 2 \text{ fm} \downarrow (P)$

Then $T =$

- 10.4 (b) Using Eq. (8.1), calculate the ratio of the number of protons having velocities ten times the rms value to those moving at the rms velocity.

$$(8.1) \quad n v dv = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-mv^2/2kT} 4\pi V^2 dv = \frac{\# \text{ of particles}}{\text{per unit vol with speed}}$$

Maxwell-Boltzmann distribution function.

between V and $(V+dv)$

$$n_v = \frac{n}{2\pi kT} \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} \frac{4\pi v^2}{4\pi} dv$$

$$n_{rms} = \sqrt{\frac{n}{2\pi kT}} \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv_{rms}^2/2kT} 4\pi V_{rms}^2 dv$$

Use $V = 10 V_{rms}$ and simplify:

$$\frac{n_v}{n_{rms}} =$$

$$N_v =$$

- 10.4 (c) Assuming (incorrectly) that the Sun is pure hydrogen, estimate the number of hydrogen nuclei in the Sun. Could there be enough protons moving with a speed ten times the rms value to account for the Sun's luminosity?

$$L_0 = \text{energy} / \text{time}$$

In 10.3 we found $N = 1.2 \times 10^{57}$ = total # of H nuclei in Sun

In 10.4a, we found $T \approx 5.6 \times 10^7 \text{ K}$ in center of Sun

Fraction of nuclei which can react $f = \frac{n_v}{n_{rms}} =$

Number of nuclei which can react $N_{\text{react}} \sim f N =$

In the proton-proton chain, $E = \gamma m c^2$ where $\gamma m = 0.76$ up (only a fraction of the mass converts to energy in fusion)

$E_{\text{total}} = N_{\text{react}} \cdot E_{\text{each}} =$

time =

- 10.5 Derive the ideal gas law, Eq. (10.11). Begin with the pressure integral (Eq. 10.10) and the Maxwell-Boltzmann velocity distribution function (Eq. 8.1).

$$(10.10) \quad P = \frac{1}{3} \int_0^{\infty} m n_v v^2 dv \quad \text{for gas of particles of mass } m$$

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$$(8.1) \quad n_v dv =$$

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$$P = \frac{1}{3} m \int_{v=0}^{\infty} v^2$$

Hint: Let $ax^2 = \frac{m}{2kT} v^2 \rightarrow v = x$
 $dv = dx$

Rewrite $P = \int_{x=0}^{\infty}$ dx

$$P =$$

Look up integral: $\int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^{n+1} a^n} \sqrt{\pi}$

For $n=2$, $\int_0^{\infty} x^4 e^{-ax^2} dx =$

Then $P =$

10.6 Derive Eq. (10.34) from Eq. (8.1)

$$\frac{(10.34)}{336} n_E dE = \frac{2n}{\pi^2} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT} dE$$

Maxwell-Boltzmann (8.1) $n_V dv =$
225

Kinetic Energy $E = \frac{1}{2}mv^2$ (non-relativistic)

$$V =$$

$$\frac{dV}{dE} =$$

Trickle: number of particles = $\frac{dn}{dv} dv = \frac{dn}{dE} dE$
 $n_V dv = n_E dE$

$$n_E dE =$$

- 10.7 Show that the form of the Coulomb potential barrier penetration probability given by Eq. (10.38) follows directly from Eq. (10.37).

(10.37) Cross section of nucleus $\sigma(E) \propto e^{-2\pi^2 U_c/E}$

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$$\text{Coulomb barrier } U_c = Z_1 Z_2 e^2 / r = 2Z_1 Z_2 e^2$$

$$\text{Kinetic energy } E = \mu m v^2 / 2 = h v$$

p. 334 reduced mass of two gas particles relative velocity between two nuclei in gas

$$E = \frac{1}{2} \mu m v^2 \Rightarrow v =$$

$$\text{Sub } v \text{ into } \frac{U_c}{E} =$$

Sub $\frac{U_c}{E}$ into (10.37):

$$\sigma(E) \propto e^{-2\pi^2 U_c/E} =$$

- 10.8 Prove that the energy corresponding to the Gamow peak is given by Eq. (10.41).

$$E_0 = \left(\frac{b k T}{2} \right)^{\frac{1}{3}}$$

PASTE Fig 10.6

Nuclear reaction rate $r = \left(\frac{2}{kT} \right)^{\frac{3}{2}} \frac{\nu_{\text{eff}} \nu_{\text{ex}}}{(\mu m_e)^{\frac{1}{2}}} \int_0^{\infty} S(E) e^{-\left(\frac{E}{kT} + bE^{-\frac{1}{2}} \right)} dE$
 (10.40) p. 339

If $S(E)$ is some slowly varying function of energy, then

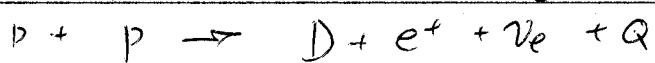
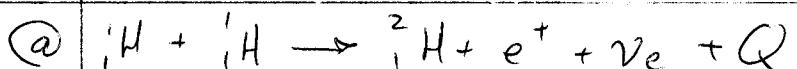
$$r = r_{\max} \text{ where } \frac{dr}{dE} = 0 \text{ and } \frac{d^2r}{dE^2} < 0$$

$$\frac{dr}{dE} = 0 \text{ when } \frac{d}{dE} \left[e^{-\left(\frac{E}{kT} + bE^{-\frac{1}{2}} \right)} \right] = \frac{df}{dE} = 0$$

Differentiate and solve:

$$\frac{df}{dE} =$$

- 10.11 The Q value of a reaction is the amount of energy released (or absorbed) during the reaction. Calculate the Q value for each step of the PP I reaction chain (Eq. 10.46). Express your answers in MeV. The masses of ${}_1^1\text{H}$ and ${}_2^3\text{He}$ are 2.0141 u and 3.0160 u, respectively.

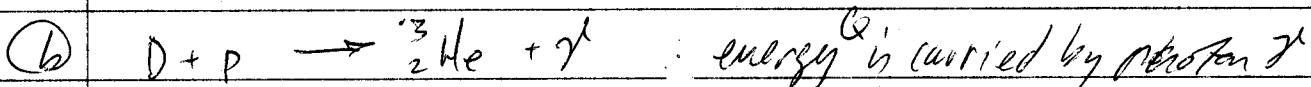


$$Q = c^2 (2m_p - m_D - m_{e^+} - m_{\nu_e}) =$$

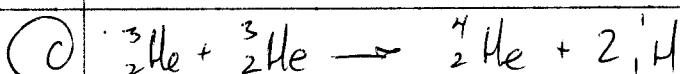
negligible

$$= c^2 (2 \cdot)$$

$$Q_{\text{PP}} =$$

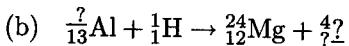
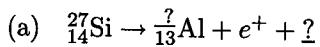


$$Q = c^2 (m_D + m_p - m_{{}_2^3\text{He}}) =$$



$$Q = c^2 (2m_{{}_2^3\text{He}} - m_{{}_2^4\text{He}} - 2m_p) =$$

- 10.13 Complete the following reaction sequences. Be sure to include any necessary leptons.



Strategy: ${}^A_Z X$ • match numbers $A = \# n + \# p$

$Z = \# \text{protons}$

• match charges ($q_p = +1 = q_{e^+} = -q_{e^-}$)

• match lepton number

③ Everything matches so $? = \gamma$ to carry off energy.

- 10.14 Prove that Eq. (10.75) follows from Eq. (10.74).

$$R = 8.31451 \times 10^{-7} \frac{\text{erg}}{\text{mole}} = \text{gas const.}$$

Ideal gas law (10.70) $PV = nRT$

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 $n = \# \text{moles per unit mass}$ (cf p. 353)

$V =$ substitute this into

(10.74) $PV^\gamma = k$ where $\gamma = \frac{C_p}{C_v}$ (10.72)

Solve for $P =$

(10.75) $P = k' T^{\frac{2}{\gamma-1}}$ so $k' =$

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- 10.15 Estimate the hydrogen burning lifetimes of stars on the lower and upper ends of the main sequence. The lower end of the main sequence²⁸ occurs near $0.085 M_{\odot}$ with $\log_{10} T_e = 3.438$ and $\log_{10}(L/L_{\odot}) = -3.297$, while the upper end of the main sequence²⁹ occurs at approximately $90 M_{\odot}$ with $\log_{10} T_e = 4.722$ and $\log_{10}(L/L_{\odot}) = 6.045$. Assume that the $0.085 M_{\odot}$ star is entirely convective so that, through convective mixing, all of its hydrogen becomes available for burning rather than just the inner 10%.

$$\frac{L_{0.085}}{L_0} = 10^{-3.297}$$

$$\frac{L_{90}}{L_0} = 10^{+6.045}$$

(a) Smallest MS stars (brown dwarfs) with $M = 0.085 M_{\odot}$ have

$$L_{0.085} = L_0 10^{-3.297} = 3.83 \times 10^{33} \frac{\text{erg}}{\text{s}} 10^{-3.297} =$$

$$L = \text{power} = \frac{\text{Energy}}{\text{time}} \text{ so lifetime} =$$

If all the star's H fuses to generate energy (an overestimate) and if the entire mass of the star is H (another overestimate) then how much energy is released by converting $H \rightarrow He$?

$$E = \Delta m c^2 \text{ where } \Delta m = 0.7 M_{\odot} \rightarrow E =$$

$$\text{For the brown dwarf, } E_{0.085} =$$

$$\text{So } T_{0.085} =$$

#10.15 b For the biggest A star, with $M = 90M_{\odot}$,

$$L_{90} = 10^{6.045} L_{\odot} =$$

$$E_{90} =$$

$$\text{So } t_{90} =$$

The bigger stars burn out much faster.

- 10.16 Using the information given in Problem 10.15, calculate the radii of a $0.085 M_{\odot}$ star and a $90 M_{\odot}$ star. What is the ratio of their radii?

Recall from Stefan Boltzmann (3.17) $L = 4\pi R^2 \sigma T^4$

Solve for $R =$

(a) $R_{0.085} =$

(b) $R_{90} =$

(c) $\frac{R_{90}}{R_{0.085}} =$

$$k = 1.38 \times 10^{-16} \frac{\text{erg}}{\text{K}}$$

$$M_0 = 2 \times 10^{33} g$$

$$R_0 = 7 \times 10^{10} \text{ cm}$$

$$\Gamma = 5.67 \times 10^{-5} \frac{\text{erg}}{\text{cm}^2 \text{sr} \text{K}^4}$$

$$a = \frac{4\pi}{c} = 7.57 \times 10^{-15} \frac{\text{erg}}{\text{cm}^3 \text{K}^4}$$

$$u_p = 1.67 \times 10^{-24} g$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} / \cancel{10^7 \text{ erg}} / \cancel{\text{J}} = 1.6 \times 10^{-12} \text{ erg}$$