

CALCULATED FICTION
MATH WORKSHOP 2

October 5, 2004

For this workshop you should work in a group with your project seminar. As before, you are to work through the workshop at the same speed as the rest of your group; stick together. For some of you, much of this workshop will be review, so this will be a good opportunity to deepen your understanding by explaining the ideas to others.

Keep your work on a separate sheet so you can turn it in at the end of the workshop.

A “Prime” Example: HOW TO TALK LIKE A MATHEMATICIAN

Read the handout entitled “How to Talk Like a Mathematician”.

ACTIVITY

1. Look back through Chapter 1 of *The Language of Mathematics* [TLOM]. Which facts from that chapter can we call theorems? Which can we call propositions? Lemmas? Corollaries? Conjectures?

TI-84, I Love You: GETTING TO KNOW YOUR CALCULATOR

Let’s begin by spending a few minutes becoming more familiar with our calculators. (If you’re already familiar with your calculator, this will be a snap; you should help the other people in your group.)

Figure out how to compute the following numbers using your calculator. Keep trying until you get the answers on the right. (Your calculator will probably show more digits than are shown here.)

$10!$	3628800
$10! / (3! 7!)$	120
35^2	1225
2^{30}	1073741824
1782^{12}	1.0254 e39 or 1.0254 +39
$\sqrt[3]{30829}$	2.366355
$2004 \bmod 139$	58

What ARE some of these things? Here’s the rundown:

$10!$ is pronounced “ten factorial” (not “TEN!”, though you are free to get that excited about it if you want to). Recall from the reading that if n is a natural number, $n!$ is the product of all of the natural numbers from 1 up through n . Thus $10! = 1*2*3*4*5*6*7*8*9*10$.

Your calculator should have a factorial button on it, so you shouldn't have to do all of that multiplication by hand – a nice feature if you have to find, say, $35!$.

2^{30} is 2 multiplied by itself 30 times. Your calculator probably has a 2^x button somewhere on it, which means you can find this in one step instead of punching in all those 2s.

Similarly, 1782^{12} is 1782 multiplied by itself 12 times. Your calculator should have an x^y button that will make finding such numbers a lot easier. The $e39$ or $+39$ part of the answer means “times 10^{39} ”. The calculator uses this when the answer is too long to fit on the screen, as it is in this case, where the answer is 40 digits long. A number written this way is said to be in *scientific notation*.

$\sqrt[12]{30829}$ is the twelfth root of 30829. That is, it's the number that, when raised to the 12th power (read: when multiplied by itself 12 times), yields 30829. Lucky for you, your calculator should have a button to do this too.

$2004 \bmod 139$ is the remainder left over when 2004 is divided by 139. Said another way, it's the result of subtracting 139 from 2004 over and over until something less than 139 is left. This is sometimes called “clock arithmetic”. If the clock says 1 o'clock now, what time will it say in 49 hours? The answer isn't 50 o'clock, right? To get the answer, you add 1 and 49 and then start subtracting 12s (50, 38, 26, 14, 2) until you get to a clock time, and that's the answer. This is the same thing, only now the clock has 139 hours on it instead of only 12. Try doing $50 \bmod 12$ on your calculator too. What about $24 \bmod 12$?

ACTIVITY

2. What's the largest factorial that your calculator can handle? That is, what's the largest natural number n such that punching $n!$ into your calculator yields an answer? What's $n!$ for the n you found, and what does the calculator say when you try to find $(n+1)!$?
3. What's the smallest factorial that your calculator can handle? What's $0!$? Can you think of a reason why your calculator's answer makes sense?
4. Find $1782^{12} + 1841^{12}$, and then take the 12th root of that. What's your calculator's answer? (This will depend on the calculator you have; your answer should be between 1920 and 1925.) Is it a whole number? If it were (or if it is), why would that be surprising?

Groin and Mousetrap: PERMUTATIONS & COMBINATIONS

How many ways are there to rearrange the letters in the word “groin”? If you reach into a bag of eight mousetraps and three of them snap shut on your fingers, how many possibilities are there for which three they were? Combinatorics – specifically the part dealing with

permutations and combinations – answers these kinds of questions for us.

A rearrangement of a set of distinct elements (like, say, the letters in “mulch”) is called a *permutation*. If we rearrange some elements, we can say we are permuting them. Hence we can permute “groin” to get the permutations “nogri” and “ringo”.

ACTIVITY

5. How many permutations of “ABC” are there? Write them down. How about “ABCD”? Write those down too.

As you can see, writing them out is not the way to find the number of permutations of a set of n elements. Luckily we have a nice formula: There are $n!$ permutations of a set of n elements. Check your answers to activity 5 against this formula; if they disagree, see which ones you forgot to write down.

What happens, though, if you want to permute only a subset of a set of n elements? For example, maybe you have 15 trophies you won at raw oyster-eating tournaments, and you’re trying to choose 6 of them to display in order on your mantel. How many ways are there to do this?

You could start by picking the leftmost one: 15 choices
Then the one to go next to it: 14 choices (since now one is gone)
Then the one to go next to that: 13 choices (now two are gone)
Then the next one: 12 choices (etc.)
Next: 11 choices
And the last (rightmost) one: 10 choices

Thus there are $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 = 3,603,600$ ways to do this.

In general, if you want to permute k elements from a set of n elements, there are

$P(n, k) = \frac{n!}{(n-k)!}$ ways to do it. To read $P(n, k)$ out loud, say “ n permute k ”. We’ll only use this formula when we care about what the actual number is; otherwise we’ll just refer to these *permutation numbers* as $P(n, k)$.

ACTIVITY

6. How many ways are there for a class of 25 students to elect a class president, vice president, and secretary? Give both an $P(n, k)$ answer and an actual number.

With permutations, order is important. But sometimes order isn’t important to us. Suppose we want to appoint a special 5-person Snack and Beverage Task Force, with the 5 people chosen from our class of 42 people. For these purposes, it doesn’t matter what order we choose them in; all that matters is who ends up on the task force. Let’s emphasize the word *choose* here: we’re choosing instead of permuting. Permutation implies order matters; choice implies order doesn’t matter.

In general, if you want to choose k elements from a set of n elements, there are

$$C(n, k) = \frac{n!}{k!(n-k)!}$$

ways to do it. To read $C(n, k)$ out loud, say “ n choose k ”. Again, we’ll refer to these *combination numbers* as $C(n, k)$ whenever possible, only using the formula when an actual number is required.

ACTIVITY

7. How many 10-person committees can we choose from the 100 people in the U.S. Senate? Give both an $C(n, k)$ answer and an actual number.
8. Calculate $C(20, 0)$. Explain why that’s the right answer.
9. Calculate $P(200, 3)$ and $C(200, 3)$.

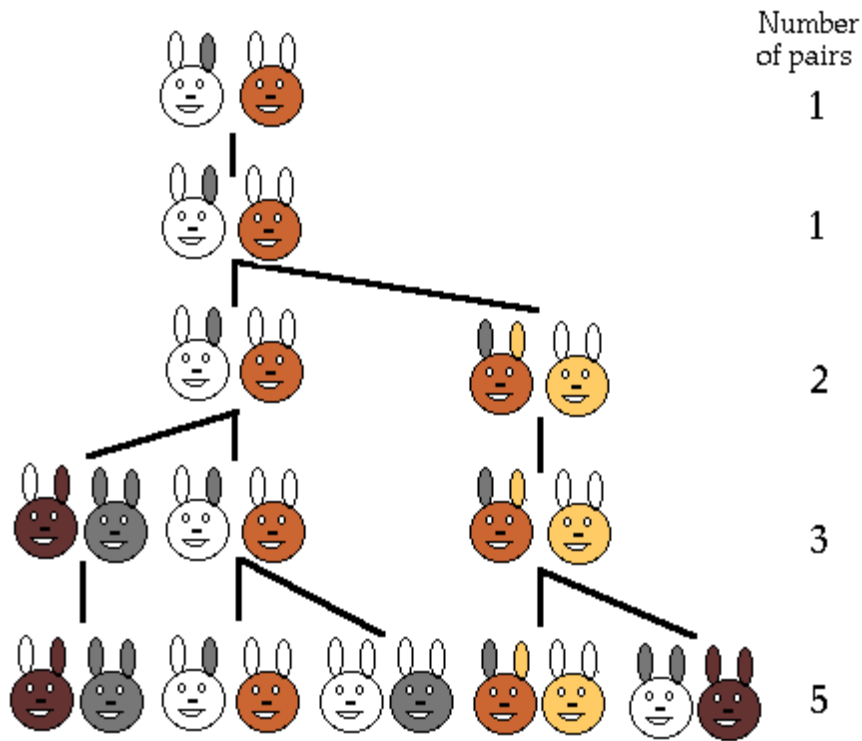
If you had trouble with that last one, there may be buttons somewhere on your calculator that will help you out; look for ${}_n C_r$, ${}_n C_k$, $C(n, r)$, or $C(n, k)$ (and similarly for P).

Breeding Like Rabbits: FIBONACCI NUMBERS

The Fibonacci numbers are named after a guy named Leonardo of Pisa (true story; Fibonacci is short for *filius Bonacci*, “son of Bonacci”). They arose from a problem that Fibonacci investigated in the year 1202 about how fast rabbits could breed in ideal circumstances. To wit:

Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. How many pairs will there be in one year?

- At the end of the first month, they mate, but there is still one only 1 pair.
- At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
- At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field.
- At the end of the fourth month, the original female has produced yet another new pair, and the female born two months ago produces her first pair also, making 5 pairs.



Continuing, the number of pairs of rabbits in the field at the start of each month is 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

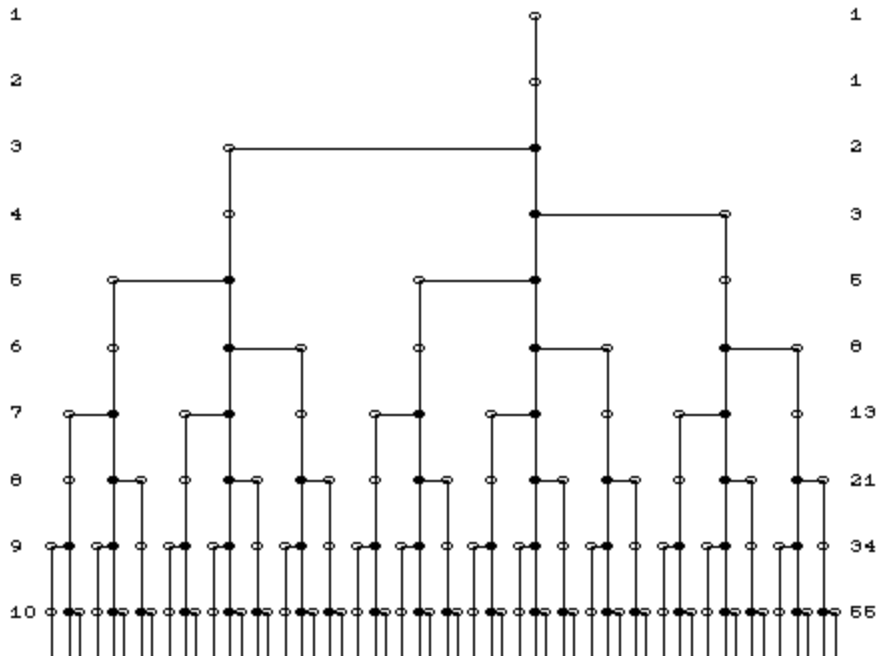
As we let the number of months stretch out forever, this defines an *infinite sequence*. Let's make a *function* out of these Fibonacci numbers: we'll say that the number of pairs of rabbits at the start of month n is $f(n)$. Thus we already know that $f(1) = 1$, $f(2) = 1$, and $f(3) = 2$. The rule for generating the rest of sequence is simple:

$$f(n) = f(n-1) + f(n-2)$$

That is, to get another number in the sequence, you just add up the previous two numbers in the sequence. Snappy!

Here's another view of the rabbit family tree. Here a vertical line represents the same pair of rabbits over time.

Family Tree of Rabbits



ACTIVITY

10. List the first 20 Fibonacci numbers.

11. Why does the above rule give the answer to Fibonacci's problem? Explain. (Think about how many rabbits survive from the previous month – which, remember, is all of them – and how many new pairs are born in month n.)

You might complain that the rabbit problem isn't very realistic, and you'd be correct. But have no fear: the Fibonacci sequence comes up all over the place in nature anyway. For example, a male honeybee has only one parent (a mother) while a female honeybee has two (a mother and a father). As a result, the number of ancestors a male honeybee has from n generations back is a Fibonacci number.

ACTIVITY

12. Draw the family tree of a male honeybee going back at least 5 generations.

13. Use your calculator to compute successive ratios of the Fibonacci numbers you listed above. That is, compute 1/1, then 2/1, then 3/2, then 5/3, then 8/5, and so on, always putting the larger one on top. Keep track of your answers. How does your sequence behave? Do the numbers seem to be heading somewhere?

$$\frac{1+\sqrt{5}}{2}$$

Compare your numbers with $\frac{1+\sqrt{5}}{2}$, the Golden Ratio (also called the Golden Number). Make a conjecture about what you see.

The Fibonacci numbers show up often in natural structures like plants (for example, the numbers of petals on a flower is often a Fibonacci number) and seashells. There a ton of fun books around that explore the Fibonacci numbers and their glorious manifestations in nature. The Golden Ratio has a long and fascinating history in art. (You may remember a discussion of it from *The Da Vinci Code*.)

But There's Only One Pointy Part: PASCAL'S TRIANGLE

For this part, you'll need to get the other handout, a photocopy of part of Pascal's Triangle.

Pascal's Triangle is named after a 17th-century French mathematician, Blaise Triangle. It's an infinite triangle of numbers. Here are the first few lines of it:

$$\begin{array}{ccccccc} & & & & 1 & & & & \\ & & & & 1 & & 1 & & \\ & & & 1 & 2 & & 1 & & \\ & & 1 & 3 & 3 & & 1 & & \\ & 1 & 4 & 6 & 4 & & 1 & & \\ 1 & 5 & 10 & 10 & 5 & & 1 & & \end{array}$$

ACTIVITY

14. From this part of the triangle, figure out how to generate the next line of the triangle. What's the general rule?

We can always use the rule to get one line of the triangle from the one before it; that's why the triangle is infinite.

Pascal's triangle has a number of great applications, many of them in the field of Combinatorics. It's also fun to play with because there are so many patterns hiding inside of it. Let's look at a few.

ACTIVITY

15. One row at a time, add up the numbers in the triangle (so you get one sum for each row). What pattern do you see? State a conjecture about it. Can you give an argument (using your rule from the previous activity, perhaps) for why your conjecture is true?

Now let's look for $C(4,0)$, $C(4,1)$, $C(4,2)$, $C(4,3)$, $C(4,4)$ in the triangle. Hey, those numbers are right there in the 4th row! (Wait – the 4th row? Yes, because here we start counting from ZERO, so the top row is row 0, the next is row 1, and so on. This may seem goofy, but it makes other things work out nicely.) Lo and behold, Pascal's triangle is made up entirely of combination numbers! This is great news for those of us who don't like calculators; all the combination numbers we want are at our fingertips, the result of a few simple additions. Of course, this isn't practical for large combination numbers.

ACTIVITY

16. Use Pascal's triangle to find $C(8,4)$. In general, where in the triangle do you look to find $C(n,k)$?

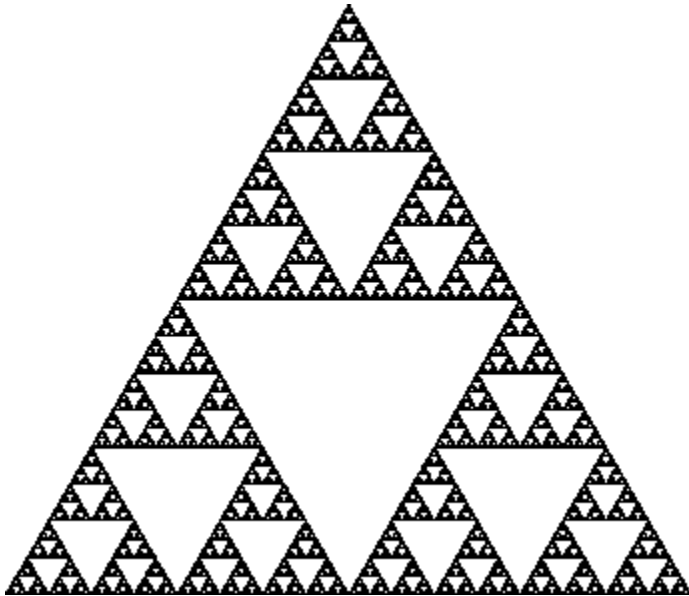
17. We can even find the Fibonacci numbers inside of Pascal's triangle, if we're willing to do a little bit of addition. Experiment with adding along diagonals until you start to see them. Be careful about which diagonals you choose; if you go too steep, you'll be adding infinitely many numbers. On a diagram, show the diagonals you followed.

Somewhat mind-blowing is the pattern you get when you just look at the multiples of 2 (which is to say, the even numbers).

ACTIVITY

18. On the handout, shade in everything EXCEPT the multiples of 2. On another copy, shade in everything except the multiples of 3. What patterns do you see?

It turns out that if you darken the odd numbers in Pascal's triangle (well, in some cut-off part of it), you get a truncated version of the Sierpinski Gasket, a figure with **zero area** obtained by successively removing the middles of equilateral triangles. Funtastic!



SOURCES

Materials for this workshop were taken and/or adapted from the following sources:

Discovering Number Theory, Jeff Holt and John Jones, 2001.

<http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html>, Ron Knott's web site on the Fibonacci numbers and the Golden Ratio.

PROJECT

1. Finish any workshop activities that you didn't get to during workshop time.

For the rest of this week's project, work on your own or with at most one other person. As always, say who you worked with.

2. Notice that each row of Pascal's triangle reads the same left to right as right to left. Since the entries in Pascal's triangle are combination numbers, this corresponds to a useful fact about $C(n,k)$. Find a way to state this useful fact as a conjecture. (To get you started, look at $C(5,0)$, $C(5,1)$, $C(5,2)$, $C(5,3)$, $C(5,4)$, and $C(5,5)$. Which ones are equal? What's the pattern?)

Once you have a conjecture, give an argument that shows it to be true. (You could use the formula for $C(n,k)$, or you could use things you know about Pascal's triangle, or you could just think about the meaning of $C(n,k)$.)

3. Continue to play with Pascal's Triangle. Find another pattern in it, or take one of the patterns discussed in the workshop. Come up with a reason that explains why your pattern is there.

4. Continue to play with the Fibonacci numbers. Do something to them that yields an interesting new pattern. For example, if you square consecutive Fibonacci numbers and add them up, it looks like you get another Fibonacci number: $1^2 + 2^2 = 5$, $2^2 + 3^2 = 13$, and $3^2 + 5^2 = 34$. This may take some doing, so be patient and keep your eyes peeled. Some other things you might try are adding up groups of consecutive Fibonacci numbers and multiplying different Fibonacci numbers together.

Once you've found a pattern, state it as a conjecture. (If you have trouble finding your own pattern, you can use the example from the previous paragraph, but you need to give evidence that you made a significant effort to find your own.)

BONUS: Use induction (discussed in TLOM) to prove your conjecture.