**CALCULATED FICTION**

# MATH WORKSHOP 3

October 12, 2004

**KEEP YOUR WORK ON A SEPARATE SHEET OF PAPER. TURN IN YOUR WORK AT THE END OF THE WORKSHOP TIME. THIS MEANS YOU.**

Work in a group with your project seminar. Work at the same speed as the rest of your group. Help each other out. Beep.

## PART THE FIRST: PROPOSITIONAL LOGIC

### "Good Sir, Thy Proposition Doth Make Me Quiver"

A *statement* or *proposition* is a declarative sentence that is either true or false, but not both. (This is the sense in which "proposition" is used on p. 63 of *The Language of Mathematics*.) We don't have to know which truth value (i.e., which of True and False) a proposition gets; we simply have to know that it must have one or the other. Consider the following examples. Some of them are propositions, and some are not.

1. In 1492 Columbus sailed the ocean blue.
2. My doctor said "Mylanta".
3. Tell Brenda to stop her noisy crocheting.
4. The pyramids were a bunch of huge triangular cubes.
5. Would you please pass the oysters?
6. $2 * 9 = 29$
7. Nobody doesn't like Sara Lee.
8. Donuts make my brown eyes blue.
9. $x + 5 = 22$
10. Over a million years before life first appeared on earth, a one-kilogram rock in the shape of a fir tree flew into the sun.

It's important to distinguish between a sentence not being true or false and us not knowing whether a sentence is true or false. For example, we probably can't tell whether the sentence "Einstein once had a dream about gargling with elephant saliva." is true or false, but the fact is that it *is* either true or false.

*ACTIVITY*

1. Discuss the above examples with your group. Decide which of these are propositions and which are not. For those that are, say if they're true or false or if we don't know. For those that are not, say what prevents them from being propositions.

You may have found yourself saying things like "It depends on how you interpret 'life'" for some of these examples. That's OK. When it really, really matters, we can be as exact as we need to be about what proposition is being expressed, especially if it's mathematical (since in that case there's bound to be some very exact notation around).

The possible surprise in the above list is #9, which mathematicians do *not* consider to be a proposition. Why not? Its truth depends on the value of x. Thus we can't say that it actually is true or false; it is simply *potentially* true or false. Note that none of the other examples have this problem. In general, a proposition can't have variable-like things in it. For example, the truth or falsity of "I've never seen one of *those* before." depends on what "*those*" refers to, so it's not a proposition. (It's not always clear whether something is a proposition or not. Luckily, when the sentence in question is a mathematical one, it generally *is* clear.)

It seems, then, that we can boil propositions down to just two possibilities: true (T) and false (F). Although they may express complicated ideas, at the end of the day they're either T or F and that's that. Following Boole (discussed in Ch. 2 of TLOM, natch), we introduce *propositional variables*, letters that stand in for propositions. Really we can think of them as just standing in for either T or F, but we don't know which. We tend to use *p*, *q*, *r*, *s*, and *t* for propositional variables.

### Compounding The Problem

Part of the point of all this propositional blahdyblah is to provide a means of representing patterns of reasoning. Certain patterns turn up again and again in *compound propositions*, propositions that are built out of other propositions. For example, "Abernathy is drunk and the pot roast is on fire" is built from "Abernathy is drunk" and "The pot roast is on fire" in the same way that "$5 < 3$ and 19 is prime" is built from "$5 < 3$" and "19 is prime". Abstracting the pattern from these examples, we can say that if *p* is a proposition and *q* is a proposition, then *p and q* is another proposition.

But how do we say what "and" means? In the propositional context, since we only have to worry about T and F as possible values for *p* and *q*, we can completely specify the meaning of "and" by saying what it does with each possible combination of inputs of T and F. To wit:

| *p* | *q* | *p and q* |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

That is, *p and q* is only true when *p* is true and *q* is also true; it's false otherwise. Note that the four listed combinations of T and F cover all the possibilities.

The above table is called the *truth table* for and. (It also appears on p. 64 of TLOM, along with the truth tables for *or*, *not*, and ®.) Given any compound proposition, we can build a truth table for it by looking at the truth tables of its constituent parts. Consider the compound proposition *(p ® q) and p*:

| *p* | *q* | *p ® q* | *(p ® q) and p* |
|---|---|---|---|
| T | T | | |
| T | F | | |
| F | T | | |
| F | F | | |

How do we fill in this table? Let's go column by column. To get the info for the *p ® q* column, we look at the truth table for *p ® q* (on p. 64 of TLOM): this tells us that that column should have T, F, T, T in it. (Write those in above.) Now how do we fill in the last column? Let's go line by line.

- For the first line, *(p ® q) and p* is the same as T *and* T; we just look to the left to see what the values of *p* ® *q* and *p* are. Thus the value for this line is the value of T *and* T, which is T. (Why?)

- For the second line, looking to the left again, we see that *(p ® q) and p* is the same as F *and* T. From the table for *and*, we see that F *and* T is F, so that's what goes here.
- For the third line, *(p ® q) and p* is the same as T *and* F, which is F.
- For the fourth line, *(p ® q) and p* is the same as T *and* F, which is F.

Whew! Now, what does this table do for us? It tells us the entire meaning of the proposition *(p ® q) and p*. Which is to say, it makes it so that if we know the truth values of *p* and *q*, then we know the truth value of *(p ® q) and p*. In particular, notice that the truth values for *(p ® q) and p* exactly match the truth values for *p and q*; this means that they have *the same meaning*. Shazam.

*ACTIVITY*

2. Make a truth table for *q or not p*. Be sure that you list the possibilities for *p* and *q* in the same order as in the above examples (T T, T F, F T, F F).

3. Think about the English word *nor*. This word can be used to combine propositions, just like *and* and *or* can. Make a truth table for *p nor q*. (If you like, think of it as "*neither p nor q*".)

### Implication Seems To Be The Hardest Word

Take another look at the truth table for implication (®). It may seem funny that we say that *p* ® *q* is true whenever *p* is false. Let's explore that a bit by looking at an example.

Suppose you go to interview for a job at a store and the owner of the store makes you the following promise: "If you show up for work Monday morning, then you will get the job." This has the form *p* ® *q*, where *p* is "you show up for work Monday morning" and *q* is "you will get the job". Under what circumstances would you say that the owner spoke falsely?

- If you show up on Monday and you get the job, she told the truth.
- If you show up on Monday and you *don't* get the job, she lied.
- If you don't show up on Monday, she told the truth (at least, you can't accuse her of lying); you blew it by not showing up.

Hence the only way that *p* ® *q* can be false is if *p* is true and *q* is false. In particular, if *p* is false, then *p* ® *q* isn't even relevant; in that case we can't say it's false, so it must be true.

Now look back at your answer for #2. The truth table (if you did it right) matches the truth table for *p* ® *q*! That means that *p* ® *q* and *q or not p* have <u>the same meaning</u>. This is also probably a surprise, if you think about it. It means, for example, that the store owner's sentence is logically equivalent to "You will get the job or you won't show up for work Monday morning." That's not how anybody would say that, but knowing that these two are equivalent can come in handy when we try to reason about implication statements.

Yet another proposition that's equivalent to *p* ® *q* is *not q* ® *not p*. Imagine that a person in a psychology experiment is trained to know that when the green light flashes, he gets a food pellet. He might say, "If the green light is flashing, then I'm getting a food pellet." Now, if he isn't getting a food pellet, could the light be flashing? No, because if it were then he'd be getting food. Hence he could just as well say, “If I’m not getting a food pellet, then the green light isn’t flashing.”

These different ways to write *p* ® *q* may be a bit uncomfortable at first, but it’s vital that we agree on how implication works so that we’re sure we’re talking about exactly the same thing when we say, for example, “If n is a positive integer greater than 2, then $x^n + y^n = z^n$ has no solutions in positive integers.” We can’t tolerate ambiguity of expression when we have to be sure we’re right.

*ACTIVITY*

4. For each of the following, translate the given statements using propositional variables and *and*, *or*, *not*, and ®. Say what your propositional variables stand for. Then decide whether or not the given pattern of reasoning is valid. That is, given the assumptions, is it correct to draw the given conclusion?

   (a) If Bessie is a cow, then she moos.
   Bessie is not a cow.
   Therefore she doesn’t moo.

   (b) If the phone rings in the next minute then I will turn into a gerbil.
   I will not turn into a gerbil.
   Therefore the phone will not ring in the next minute.

   (c) If I go to the party then I cannot do my homework.
   I did not do my homework.
   Therefore I went to the party.

   (d) If I work, then I will have plenty of money.
   If I don’t work, then I have a good time.
   Therefore, I have either money or a good time.

## PART THE SECOND: PROOF

### The Pudding Is In The Proof

This work to understand compound propositions is for a good cause: it helps us to write and understand proofs. But why are proofs such a big deal?

One nice thing about Mathematics is that it allows us a high degree of certainty. We can be certain, for example, that the Pythagorean Theorem is true, without having to check the sides of each right triangle we meet to make sure they follow the correct pattern. Proof is what enables us to have that certainty; it’s because we have a *proof* of the Pythagorean Theorem that we can be fully confident of its truth. Proof is the primary tool of the pure mathematician; it’s her means of investigating the mathematical world. Without a solid logical foundation on which proofs can stand, the whole enterprise would lose its validity, and our certainty in mathematical results would fail. Thus when absolute truth is on the line, it’s important for us to be unambiguous and to communicate as clearly and exactly as possible. That’s why we need to spend time, for example, talking about *exactly* what we mean by implication.

*ACTIVITY*

5. With your group, spend a few (5-10) minutes discussing why certainty in mathematical truths is important and/or desirable. Jot down a couple of the reasons you discuss.

### Such another proof will make me cry ‘baa.’

- Speed, in Wm. Shakespeare's *The Two Gentlemen of Verona*

On p. 51 of TLOM, Devlin says a proof is something that has "the capacity to completely convince any sufficiently educated, intelligent, rational person." That's a high standard! We're not going to go through the training required to become professional mathematicians right now, so we don't have to write proofs that meet that standard, but we can still do our best to write convincing logical arguments about mathematical ideas. Let's agree for now that "proof" will simply mean "convincing argument". We have some tools around – for example, propositional logic – to help us as we construct these arguments.

One of the best ways to understand how proofs work is to study examples. Consider these proofs that the sum of an odd number and an even number is odd:

**Theorem**: The sum of an odd number and an even number is odd.

*1st Proof.* Let $m$ be odd and let $n$ be even. That means that $m = 2r + 1$ for some natural number $r$ and $n = 2s$ for some natural number $s$. Then we have

$$\begin{aligned} m + n &= (2r + 1) + 2s \\ &= 2r + 1 + 2s \\ &= 2r + 2s + 1 \\ &= 2(r + s) + 1 \end{aligned}$$

This is an even number ($2(r + s)$) plus 1, so it is an odd number. ♦

(It's common to use a special symbol to denote the end of a proof; we'll use ♦.)

*2nd Proof.* Let $m$ be odd and let $n$ be even. Suppose we have a pile of $m$ stones and a pile of $n$ stones. Since $m$ is odd, if we pair up the stones in the first pile, there will be one left over. Since $n$ is even, if we pair up the stones in the second pile there won't be any left over. Now, addition of $m$ and $n$ is the same as putting the two piles together. When we do that, the stones will still be paired up as they were, but there will still be one left over. Hence there are an odd number of stones, which means $m + n$ is odd. ♦

Notice that although we *can* use highly mathematical language and notation to write a proof, we don't *have* to. The second example is no less a proof than the first one is, even though it uses a less obviously mathematical approach. Think of proofs as simply being good arguments written about mathematical statements.

*ACTIVITY*

6. What facts and definitions are being assumed in the 1st proof above?

7. Write a proof that the sum of two odd numbers is even.

8. Come up with a proof that addition of natural numbers is commutative, that is, that $m + n = n + m$ for all natural numbers $m$ and $n$. Have someone play devil's advocate; they should challenge you to explain why each thing in your proof is true.

## PART THE THIRD: PROBLEM-SOLVING

### Wherein Logic, Proof, and Good Times Share A Nice Cup Of Tea

Solving logic puzzles (such as the ones below) uses many of the same skills as writing proofs. If you find the answer, you should be able to show that it's correct, right? And showing it's correct is essentially writing a proof. As you work through these brainteasers, the approach of propositional logic should be useful to you. That is, considering possible combinations of truth and falsity will often lead you to the answer.

*ACTIVITY*

9. Consider a silver and a golden pot, one of them containing a treasure, the other empty. Assume that you can determine from the writing on the pots which one contains the treasure. The inscriptions on the pots are:

   The silver pot: "This pot is empty."
   The golden pot: "Exactly one of these texts is true."

   Which pot contains the treasure?

10. You're traveling on a magical island, trying to find the castle. You come to a fork in the road; you know that one fork leads to the castle and the other fork leads away from it. There are two dudes standing at the fork; one of them always tells the truth and the other one always lies. You know this about them, and they know it about each other, and they both know which fork leads to the castle, but unfortunately you don't know which of them is the truth-teller and which of them is the liar.

    By asking only one of the dudes only one question, how can you determine which fork leads to the castle?

11. (a) What yes/no question could I ask you that, in answering, it would be logically impossible for you to lie? (I'm assuming that you answer either yes or no.)

    (b) What yes/no question could I ask you that, in answering, it would be logically impossible for you to tell the truth? (Ditto.)

12. In a small village in the middle of nowhere, three innocent prisoners are sitting in a jail. One day, the cruel jailer takes them out and places them in a line on three chairs, in such a way that man C can see both man A and man B, man B can see only man A, and man A can see none of the other men. The jailer shows them 5 hats, 2 of which are black and 3 of which are white. After this, he blindfolds the men, places one hat on each of their heads, and removes the blindfolds again. The jailer tells his three prisoners that if one of them is able to determine the color of his hat within one minute, all of them are released. Otherwise, they will all be shot. None of the prisoners is able to see his own hat at any point, no-one moves, and all are intelligent. After 59 seconds, man A shouts out the (correct) color of his hat!

    What is the color of man A's hat, and how does he know?

13. Ms. X, Ms. Y, and Ms. Z - an American woman, an Englishwoman, and a Frenchwoman, but not necessarily in that order, were seated around a circular table, playing a game of hearts. Each passed three cards to the person on her right. Ms. Y passed three hearts to the American. Ms. X passed the queen of spades and two diamonds to the person who

passed her cards to the Frenchwoman.

Who was the American? The Englishwoman? The Frenchwoman?

14. Consider these sentences:

(1) At least two sentences in this list are true.
(2) At least two sentences in this list are false.
(3) At least one sentence in this list is false.

Which sentences are true and which are false?

15. Consider the following two sentences:

A: Sentence B is false.
B: Sentence A is true.

What are the truth values of A and B?

## REFERENCES

Some of the materials for this workshop were taken/adapted from these sources:


- *Discrete Mathematics with Applications*, Second Edition, by Susanna S. Epp
- *Discrete Mathematical Structures*, $5^{th}$ Edition, by Kolman, Busby, and Ross
- *The Riddle of Scheherazade and Other Amazing Puzzles*, by Raymond Smullyan
- A collection of Discrete Math problems and examples from Professor Peter Henderson (Butler University)


## PART THE PROJECT

1. Finish anything from the first two parts of the workshop that you didn't get to in class. (Give some thought to the third part too, but don't feel like you have to come up with a solid solution to every problem; there may be too much here to do.)

It's nearly time to start thinking about your final project already. Toward that end, in this week's project you'll have to do some independent research to learn a mathematical topic that might be interesting enough to you to provide the foundation for your final project. **You are to work alone on this week's project.** Remember that 3 hours per week is the minimum guideline for how much time to spend on the math project.

2. Choose a mathematical topic or idea that interests you. You can choose something that has come up in this program or not, as you wish. Find a source (a book or a website) that deals with your topic and learn the topic well enough to teach it to someone else. Write a short paper, between one and two **typed** pages in length, in which you fully explain your topic or idea in your own words. Convince me that you know what you're talking about. Illustrate with your own examples. *Demonstrating thorough understanding is more important than picking an big idea*; you'll have to fully understand your topic in order to incorporate it into your writing.