CALCULATED FICTION

MATH WORKSHOP 7

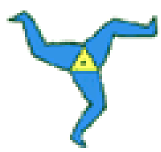
November 9, 2004

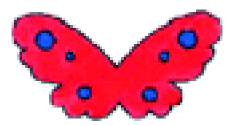
To mix things up, this time you'll work in random groups again. You need to get out more! Listen to the math guy to find out how to get into groups.

THE FEARFUL SYMMETRY OF... TIGERS? (1:20p – 2:45p)

Human beings seem to find buildings with a line of symmetry attractive, perhaps because they themselves have an approximate vertical line of symmetry. Crystals, flowers and magnified snowflakes all appeal to us too, and much of this appeal lies in their symmetry.

So, what do we mean by symmetry? Have a look at these two drawings.





Both of these two shapes can be moved in such a way that we can't tell that any alteration has been made. A movement which behaves like this is called a *symmetry transformation*.

ACTIVITY

1. How many symmetry transformations can you find for each of the above drawings? What are they?

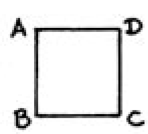
The figure on the left (which is based on the symbols of the Isle of Man and of Sicily) has *rotational symmetry* because its symmetry transformations are rotations. The butterfly, on the other hand, has no rotational symmetry; it only has *reflectional symmetry*. (Reflectional symmetry is also called *line symmetry* or *mirror symmetry* because there is a line in the figure where a mirror could be placed, and the figure would look the same.)

ACTIVITY

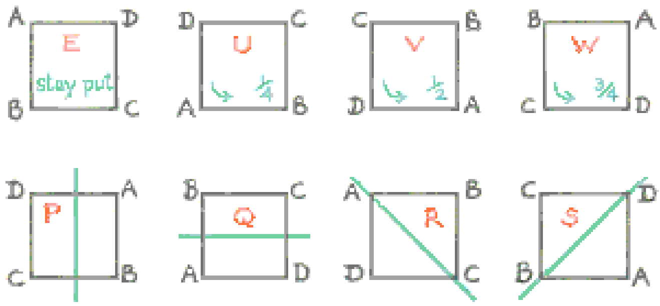
2. Which capital letters of the English alphabet have reflectional symmetry? Do any of them have more than one line of symmetry? (Feel free to make any useful assumptions about the font.)

3. Which capital letters of the English alphabet have rotational symmetry?

Squares share the same kinds of symmetry as both the butterfly and the 3-legged symbol. Consider a square with the vertices labeled as follows:



There are 8 different symmetry transformations for the square. To wit:



The rotational symmetries of the square are:

E .. This is the 'stay put' transformation. E is called the *identity*.

U .. This rotates the square through a quarter turn.

V .. This rotates the square through a half turn.

W .. This rotates the square through a three-quarter turn.

Note that we've decided that all rotations are counterclockwise. We don't need to have clockwise rotations too, because all that matters is where the figure ends up, not how it gets there. Thus a clockwise quarter turn would be redundant in our list because it's the same transformation as a counterclockwise three-quarter turn.

The reflectional symmetries are:

P .. This reflects across a vertical mirror line through the center.

Q .. This reflects across a horizontal mirror line through the center.

R .. This reflects across a diagonal mirror line through AC.

S .. This reflects across a diagonal mirror line through BD.

If you choose any pair of rotations from E, U, V and W, and do one followed by the other, then the result will also be one of E, U, V or W. For example, if we do U (a quarter turn) followed by V (a half turn), that's the same as doing W (a three-quarter turn). Let's agree to write this in equation form as V*U = W. **Notice that although we do U and then V, we write V on the left in our equation.**

We can quickly write down several more such equations, because E, the 'stay put' transformation, combined with any other transformation is the same as just doing the other transformation. Thus we see, for example, that E*U = U and U*E = U.

Let's summarize this information about combining transformations in a delicious table. When we want to say that A*B = C, we'll put A on the left-hand side of the table and B on the top; where A's row and B's column meet, we'll put in a C. (This is exactly like in a regular multiplication table.) Here's what we know so far:

	E	**U**	**V**	**W**
E	E	U	V	W
U	U			
V	V	W		

W	W			

ACTIVITY
4. Fill in the rest of the table.

You should see from your table that each rotation has a 'partner' so that together they give the stay-put transformation of E. U and W have each other, V has itself, and E is E anyway. The partner of a transformation is called its *inverse*. It's a key point that every symmetry transformation has an inverse that's also a symmetry transformation; it's the "undo whatever the original one does" transformation.

Now let's think about the reflections.

ACTIVITY
5. What do you get if you do P followed by P? P followed by R? R followed by P?

These questions could be written as: What's P*P? R*P? P*R? You should have found that P*R is NOT the same as R*P; if you didn't, go back and figure out why they're not the same.

It may surprise you that P*R ≠ R*P. After all, we can multiply numbers in whatever order we like and get the same answer, so we might expect to be able to 'multiply' transformations in any order as well. Alas, alack; we can't, as looking at the square will tell us.

Just as before, we can make a "multiplication" table for the reflections.

	P	**Q**	**R**	**S**
P				
Q				
R				
S				

ACTIVITY
6. Fill in the table for P, Q, R, and S.

Surprise! All of the entries in this table are E, U, V, or W. (If you got any P's, Q's, R's, or S's, go back and try again.)

As long as we're making tables, we may as well make one with all 8 transformations in it. The shaded parts correspond to the two smaller tables we've already made.

	E	**U**	**V**	**W**	**P**	**Q**	**R**	**S**
E								
U								
V								
W								
P								
Q								
R								

S								

ACTIVITY

7. Fill in the whole table for E, U, V, W, P, Q, R, and S.

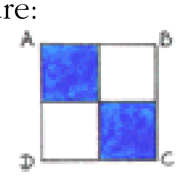
As you might recall from reading Chapter 5 of TLOM, what we've been building here is the *symmetry group* of the square. (The table for the symmetry group of an equilateral triangle appears on p. 195 of TLOM.) In particular, this is a group, which means it satisfies the group axioms (listed on p. 191). There are lots of facts that we get for free once we know something is a group. Some of the interesting ones:

- Every element appears exactly once in each row and exactly once in each column.
- The element E leaves all other elements unchanged, as noted above.
- Every element has an inverse (a partner that yields E).
- Harder to see: if we do any *three* transformations, it doesn't matter how we pair them up. That is, for example, (U*V)*W = U*(V*W). (Check it!)

It turns out that if an arbitrary table has these four properties, then it is the table of a group. So now we know exactly what group tables look like.

Now look back at the first table we made, which showed how the rotations combine together. This table also satisfies the rules for a group! It is called a *subgroup* of the larger group. In general, a subgroup is any subset of a group so that if you multiply any two elements from the subset you get another element from the subset. For example, {E, U, P} is NOT a subgroup because P*U = R and R is not in the subset.

Now let's do another one. Consider this figure:



ACTIVITY

8. How many symmetry transformations does this figure have? Just as we did for the square, make the table for this figure's symmetry group. (Hint: Since it's a square with some stuff added, its symmetry transformations must all be ones we already looked at.)

THERE'S NO "I" IN "NONLNEAR" (2:45p – 4:00p)

Now let's turn our attention to Verhulst's equation, which is discussed in Chapter 3 (the first one) of *Turbulent Mirror*. Take a few minutes to review the discussion of this equation, on pages 56-62. While you're at it, enjoy the drawing on page 57 of Alice having just eaten a gypsy moth. Yum!

In this part of the workshop, we're going to use Verhulst's equation to see some period-doubling for ourselves. We'll need our calculators and our mad phat graphing skills.

Just for ease of reference, here is Verhulst's equation: $X_{n+1} = BX_n(1 - X_n)$

Recall that B is the birthrate. X_0 is the starting population, expressed as a percentage of the total possible population (so it's a number between 0 and 1). In what follows, you'll make four graphs, corresponding to different values of B and X_0.

ACTIVITY

9. Let $B = 2.5$, $X_0 = .5$. Find X_1 through X_{20} and make a graph that shows your data. What's the overall trend?

As a check, in Activity 9 you should find that $X_1 = .625$, $X_2 \approx .5859$, and $X_3 \approx .6065$. If you're getting very different numbers, you're making a mistake; if you have trouble figuring out what the problem is, ask for help.

A Note About Rounding: As you do these computations, you may find it easiest to round the numbers you get and type them back in each time. That's fine as long as you **always keep at least 3 decimal places**. For example, if you get $X_2 \approx .5859$, you can plug in .586 when you compute X_3.

A Note About The Division Of Labor: You may want to divvy up the computational work for these activities among the members of your group. That's fine *as long as* you all compare and discuss your results once the computations are done (and as long as you all make your own copies of the resulting graphs).

ACTIVITY

10. Let $B = 2.5$, $X_0 = .1$. Before you compute anything, what do you expect the trend to be? Now find X_1 through X_{20} and make a graph that shows your data. How does it compare with the previous graph?

11. Now let $B = 3.1$, $X_0 = .5$. Make a graph of X_1 through X_{20} again. What's the trend this time?

12. Now let $B = 3.5$, $X_0 = .5$. Make a graph of X_1 through X_{20} again. What's the trend this time?

You should find that your data points split into four groups in this last one. The diagram on page 61 of Turbulent Mirror predicts that; it even tells you what the four repeating values should be. The chaotic mess that occupies most of that diagram tells you what happens at other values of B.

ACTIVITY

13. Use the diagram on page 61 to find a value of B that would make your data split into 3 groups. Using $X_0 = .5$ again, find and graph X_1 through X_{20} and see that you chose correctly.

REFERENCES

Much of this workshop was shamelessly borrowed from the following sources:

http://www.netcomuk.co.uk/~jenolive/homegrps.html
http://www.geom.uiuc.edu/~demo5337/s97a/students.html

PROJECT

PART 1: WORK ON THE DRAFT OF YOUR FINAL PROJECT

Since you get math project time for this, you have to make your draft *extra* mathy. Pack in as much math as you can handle. Then pack in 37% more math flavor. Alive with pleasure!

Seriously, math needs to provide the skeleton, nervous system, and circulatory system for your project. Maybe also the pancreas.

PART 2: CHAOS, COMPUTERS, AND THE MEANING OF LIFE

This part is due next Tuesday, November 16, at the beginning of the math workshop.

Download the Excel file CFmathproj7.xls from the course webpage. (It will be available by Wednesday afternoon.) Follow the instructions on the instructions sheet. Treat yourself to a nice egg cream.

OH NO, IT'S A

POP QUIZ

November 9, 2004

This is a **closed-book**, **open-note** quiz on the reading from TLOM for this week. Answer each question to the best of your ability; fuzzy or intuitive answers are OK, as are examples in lieu of answers. You have 10 minutes to complete this quiz.

0. Have you read Chapter 5 of TLOM?

1. What is a group, and what do groups have to do with symmetry?

2. What is a tiling of space?

3. Circle the items from the list below that are discussed in Chapter 5 of TLOM.

Designing the ideal fence

How ink dries

Snowflakes

When to buy & sell stocks

Crystallography

Wallpaper patterns

Chinese checkers

Regrettable luncheons
How to win at blackjack

Lattices

Rhombic peas

Ice cubes

Designing cereal boxes

How to stack oranges

Honeycombs

Automobile radiators