Astrophysics Ch 2 #1, 2, 7, 8, 11 Thus 5 Zta

due ~~Tues 3~~ Feb 04

2.1 Assume that a rectangular coordinate system has its origin at the center of an elliptical planetary orbit and that the coordinate system's x axis lies along the major axis of the ellipse. Show that the equation for the ellipse is given by

The hard way;

(2.1) 28 $r + r' = 2a$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

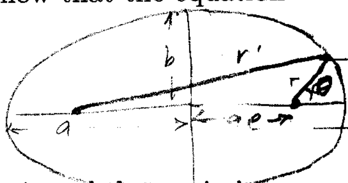
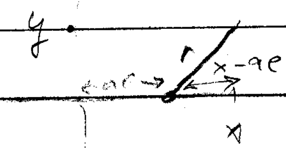


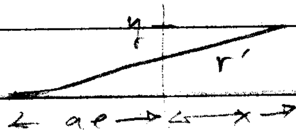
Fig. 2.4

where a and b are the lengths of the semimajor axis and the semiminor axis, respectively.

(2) $r^2 = y^2 + (x - ae)^2$



(3) $r'^2 = y^2 + (\cancel{x} + ae)^2$



$r + r' = 2a$ $\qquad e = \dfrac{\sqrt{a^2-b^2}}{a}$ $\qquad e^2 = \dfrac{a^2-b^2}{a^2}$

$\left(1 - e^2 = \dfrac{a^2 - (a^2 - b^2)}{a^2} = \dfrac{b^2}{a^2}\right)$

$r + r' = 2a = r + \sqrt{y^2 + (x+ae)^2}$

$(2a - r) = \sqrt{y^2 + (x+ae)^2}$

$(2a - r)^2 = y^2 + (x+ae)^2 = 4a^2 - 4ar + r^2$

$\cancel{y^2} + (x+ae)^2 - 4a^2 = r^2 - 4ar$

$= \cancel{y^2} + (x-ae)^2 - 4a\sqrt{y^2 + (x-ae)^2}$

$(\cancel{x^2} + 2aex + \cancel{a^2e^2}) 4a^2 = (\cancel{x^2} - 2aex + \cancel{a^2e^2}) - 4a\sqrt{y^2 + (x-ae)^2}$

$$2aex \quad -4a^2+2aex = -4a\sqrt{y^2+(x-ae)^2}$$

$$4aex \quad -4a^2 = -4a\sqrt{y^2+(x-ae)^2}$$

$$-ex + a = \sqrt{y^2+(x-ae)^2}$$

$$(a-ex)^2 = a^2+e^2x^2-2aex = y^2+(x^2-2aex+a^2e^2)$$

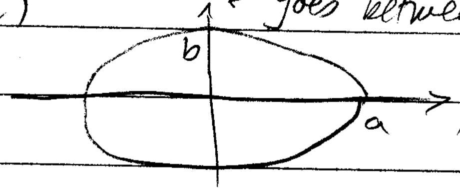
$$a^2+e^2x^2 = y^2+x^2+a^2e^2$$

$$y^2+x^2(1-e^2)=a^2(1-e^2)$$

$$y^2+x^2\left(\frac{b^2}{a^2}\right)=a^2\left(\frac{b^2}{a^2}\right)=b^2 \rightarrow \boxed{\frac{y^2}{b^2}+\frac{x^2}{a^2}=1}$$

2.2 Using the result of Problem 2.1, prove that the area of an ellipse is given by $A = \pi ab$.

$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$ y goes between $\pm b\sqrt{1-\frac{x^2}{a^2}}$



x goes from (-a to a)

$$\text{Area} = \iint dA = \int_x\int_y dy\,dx = \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{+b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$$

$$= \int_{x=-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\,dx = 2b\int \cos\theta\,dx = 2ba\int\cos^2\theta\,d\theta$$

$x = a\sin\theta \rightarrow dx = a\cos\theta\,d\theta$

$$= \frac{2ba}{2}\int(1+\cos 2\theta)\,d\theta = ba\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= ba\left(\left(\frac{\pi}{2}+\frac{\pi}{2}\right)+\frac{1}{2}\left(\sin\pi-\sin(-\pi)\right)\right)$$

$$= ba(\pi+0)$$

$$A = ba\pi$$

U1 U2 U5 U19 A3

Matt

2.11 Cometary orbits usually have very large eccentricities, often approaching (or even exceeding) unity. Halley's comet has an orbital period of 76 yr and an orbital eccentricity of $e = 0.9673$.

(a) What is the semimajor axis of Comet Halley's orbit?

(b) Use the orbital data of Comet Halley to estimate the mass of the Sun.

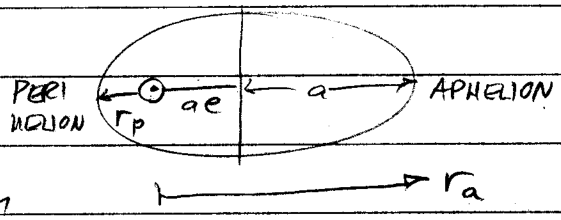
(a) $a^3_{(AU)} = P^2_{(yr)} \Rightarrow a = P^{2/3} = 76^{2/3} = \underline{17.9}$ AU $\left| \frac{1.5\times10^{11}\,m}{AU} \right|$

$a_{Halley} = \underline{2.7\times10^{12}\ m}$

(b) $M_\odot \simeq \frac{4\pi^2 a^3_{(m)}}{G P^2_{(sec)}} = \frac{4\pi^2 (2.7\times10^{12}\ m)^3}{(76\,yr \times \pi\times10^7\,\frac{sec}{yr})^2\, 6.67\times10^{-11}\,\frac{m^3}{kg\cdot s^2}} = \underline{2\times10^{30}}$ kg ✓

(c) Calculate the distance of Comet Halley from the Sun at perihelion and aphelion.

$r_{perihelion} = a - ae = a(1-e)$
$= (1-0.9673)(2.7\times10^{12}\ m)$
$= \underline{8.8\times10^{10}}\ m$



$r_{aphelion} = a + ae = a(1+e) = (1.09673)(2.7\times10^{12}\ m)$
$= \underline{5.3\times10^{4}}\ m$

U1 U2 U5 U19 A3

$e = 0{,}9673$

2.11 (d) Determine the orbital speed of the comet when at perihelion, at aphelion, and on the semiminor axis of its orbit.

$\frac{(2.30)}{50}$ $V_p^2 = \frac{GM}{a}\left(\frac{1+e}{1-e}\right)$ $\frac{(2.31)}{51}$ $V_a^2 = \frac{GM}{a}\left(\frac{1-e}{1+e}\right)$

$\frac{GM}{a} = \frac{6{,}67\times10^{-11}\frac{m^3}{kg\,s^2}(2\times10^{30}\,kg)}{(2.7\times10^{12}\,m)} = \underset{10}{4.9\times10^{7}}\ \frac{m^2}{s^2}$

$(1+e) = \underset{7}{1{,}96730}$ $(1-e) = \underset{14}{0{,}03270}$

$V_p^2 = \frac{1{,}96730}{0{,}03270}\left(4{,}9\times10^{7}\ \frac{m^2}{s^2}\right) \to V_p = 5{,}5\times10^{4}\ \frac{m}{s}$ ✓

$V_a^2 = \frac{0{,}03270}{1{,}96730}\left(4{,}9\times10^{7}\ \frac{m^2}{s^2}\right) \to V_a = 9\times10^{2}\ \frac{m}{s}$

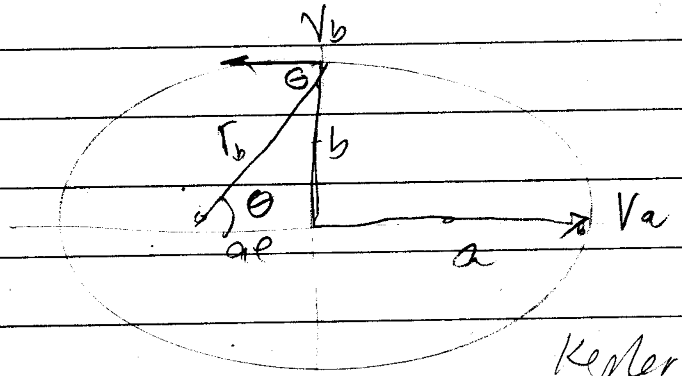
(e) How many times larger is the kinetic energy of Halley's comet at perihelion when compared to aphelion?

$K_p = \frac{1}{2}mV_p^2$

$K_a = \frac{1}{2}mV_a^2$

$\frac{K_p}{K_a} = \frac{V_p^2}{V_a^2} = \frac{\left(\frac{1+e}{1-e}\right)}{\left(\frac{1-e}{1+e}\right)} = \frac{(1+e)^2}{(1-e)^2}$

$\frac{K_p}{K_a} = \left(\frac{1{,}96730}{0{,}03270}\right)^2 = 3{,}6\times10^{3}$ ✓

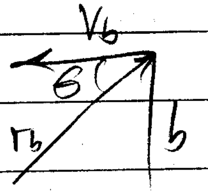


Kepler's 2d Law:

(d) V_b at semi minor axis ? $L_a = L_b$

$m V_a a = m V_b r_b = m V_b b$

$V_a a = V_b b$



$\bar{V}_b \times \bar{r}_b = V_b b$

$e^2 a^2 = a^2 - b^2$ so

$b^2 = a^2(1-e^2)$

$\left(\frac{b}{a}\right)^2 = (1-e^2) = 1 - 0.9673^2 = 0.0643$

$\frac{b}{a} = 0.2536$

$V_b = V_a \left(\frac{a}{b}\right) = 9\times10^2 \frac{m}{s} (3.9427) = \underline{3.55\times10^3}$

That's reasonable. It's between V_a & V_p