Phys-B - week 6 - Giancoli Ch 11 #10, 37, 38

Raff Ch. 11.1.1 #2, 4, 5, 6

10. (II) A particle is located at $\mathbf{r} = (4.0\mathbf{i} + 8.0\mathbf{j} + 6.0\mathbf{k})$ m. A force $\mathbf{F} = (16.0\mathbf{j} - 4.0\mathbf{k})$N acts on it. What is the torque, calculated about the origin?

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 8 & 6 \\ 0 & 16 & -4 \end{vmatrix} = 4 \times 4 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & \frac{3}{2} \\ 0 & 4 & -1 \end{vmatrix}$$

$$\tau_x = 4^2\left[2(-1) - \tfrac{3}{2}(4)\right] = 16[-2-6] = -128 \text{ N·m}$$
$$\tau_y = -4^2\left[1(-1) - \tfrac{3}{2}(0)\right] = -16[-1] = +16 \text{ N·m}$$
$$\tau_z = +4^2\left[1(4) - 2(0)\right] = 16[4-0] = +64 \text{ N·m}$$

(NOT REQUIRED)

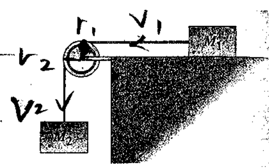


FIGURE 11–25
Problems 25 and 26.

25. (II) Figure 11–25 shows two masses connected by a cord passing over a pulley of radius R_0 and moment of inertia I. Mass M_1 slides on a frictionless surface, and M_2 hangs freely. Determine a formula for (*a*) the angular momentum of the system about the pulley axis, as a function of the speed v of mass M_1 or M_2, and (*b*) the acceleration of the system.

Notice that, as in our Labs week 6, we can calculate angular momentum of masses moving linearly:

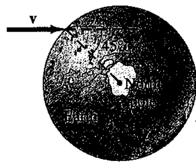
$\vec{L} = \vec{r} \times m\vec{v} = rmv$ (since $r \perp v$)

$L_1 = m_1 r_1 v_1$ and $L_2 = m_2 r_2 v_2$ where
$r_1 = r_2 =$ radius of pulley and $v_1 = v_2$ changes in time.

Total angular momentum $L = L_1 + L_2$ is NOT CONSERVED since there is an external TORQUE $= \vec{r} \times \vec{F}$.

TORQUE $= \vec{r} \times \vec{F} = r_2 \times (mg - T)$
T tension in string

$\vec{\tau} = \frac{d\vec{L}}{dt}$



Earth turns with ω_0. Angular momentum $L_0 = I\omega_0$

$m =$ **FIGURE 11–27** Problem 37.

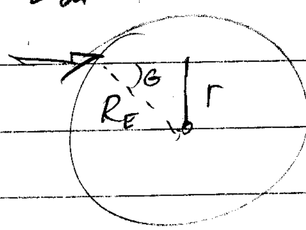
37. (II) Suppose a 7.0×10^{10} kg meteor struck the Earth at the equator with a speed $v = 1.0 \times 10^4$ m/s, as shown in Fig. 11–27 and remained stuck. By what factor would this affect the rotational frequency of the Earth (1 rev/day)? $= \omega_0$

Since $m_{meteor} \ll M_{Earth}$, I does not change much.

moment of inertia of Earth

The angular momentum of the meteor at impact is

$L_m = |m\vec{r} \times \vec{v}| = m R_E \sin\theta\, V$ $\quad \sin 45° = \frac{\sqrt{2}}{2}$



$L_{before} = L_{after}$ new speed

$L_0 - L_m = I\omega$

$I\omega_0 - mvR_E\sin\theta = I\omega$

$I(\omega_0 - \omega) = mvR_E\sin\theta$

$\omega_0 - \omega = \frac{mv}{I} R_E \sin\theta = \frac{mv}{\frac{2}{5}M_E R_E^2} R_E \sin\theta = \frac{5}{2}\frac{mv\sin\theta}{M_E R_E}$

$= \frac{5}{2}\frac{(7\times10^{10}\text{ kg})}{6\times10^{24}\text{ kg}}\frac{10^4 \frac{4}{3}\left(\frac{\sqrt{2}}{2}\right)}{6.4\times10^6\text{ m}} = 3\times10^{-17}\ \frac{\text{rad}}{\text{sec}}$

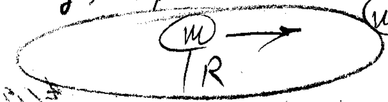
$\omega_0 = 2\pi$ rad/s

So $\frac{\omega_0 - \omega}{\omega_0} =$ small $=$ ____ %

Earth's rotation would slow by about ____

negligibly

initially, $I_{person} = 0$



ⓜ : $I_{person} = mR^2$

initial $\omega_0 \neq \omega$ final

38. (II) A person of mass 55 kg stands at the center of a rotating merry-go-round platform of radius 2.5 m and moment of inertia 670 kg · m². The platform rotates without friction with an angular velocity of 2.0 rad/s. The person walks radially to the edge of the platform. (*a*) Calculate the angular velocity when the person reaches the edge. (*b*) Compare the rotational kinetic energies of the system of platform plus person before and after the person's walk.

$$L_{before} = I\omega_0 \qquad L_{after} = I\omega + L_{person}$$

$$I\omega_0 = I\omega + mR^2\omega$$

$$\omega_0 = \omega + \frac{mR^2}{I}\omega = \omega\left(\frac{mR^2}{I} + 1\right)$$

$$\frac{I_{person}}{I_{merry go round}} = \frac{mR^2}{I} = \frac{55\,kg\,(2.5\,m)^2}{670\,kg \cdot m^2} = \underline{0.5} \text{ surprisingly high}$$

$$\left(\frac{mR^2}{I} + 1\right) = \underline{1.5}$$

ⓐ Final Speed $\omega = \dfrac{\omega_0}{\left(\frac{mR^2}{I} + 1\right)} = \dfrac{2\,rad/s}{(1.5 = 3/2)} = \underline{4/3}\ rad/s$

ⓑ $K_{before} = \frac{1}{2}I\omega_0^2 = \frac{1}{2}(670\,kg \cdot m^2)\left(\frac{2\,rad}{s}\right)^2 = \underline{1340}\ J$

$$K_{after} = \frac{1}{2}(I + mR^2)\omega^2 = \frac{1}{2}I\left(1 + \frac{mR^2}{I}\right)\omega^2$$

$$= \frac{1}{2}I\left(1 + \frac{mR^2}{I}\right)\left[\frac{\omega_0}{\left(\frac{mR^2}{I} + 1\right)}\right]^2 = \frac{\frac{1}{2}I\omega_0^2}{\left(\frac{mR^2}{I} + 1\right)}$$

$$K_{after} = \frac{K_{before}}{\left(\frac{mR^2}{I} + 1\right)} = \frac{1340}{1.5} = 893\ J$$

Person walking did work to slow merry-go-round.

E/2

(FORCE UNITS:)

Calf Cu/1.1.1 #2, 4, 5, 6

SI: $N = kg\ m/s^2$

11.2 In three-dimensional space, a classical particle moves in a potential field given by

$$u = V(x, y, z) = ax^3 + by^3 + cz^3 + dxy + exz + fyz,$$

CGS: $dyne = g \cdot cm/s^2$

where $a, b, c, d, e,$ and f are constants.

(A) When the particle is at the point $x = 1$ cm, $y = 2$ cm, and $z = 3$ cm, what is the x component of force acting on it in terms of the constants of the problem?

$$\vec{F} = -\vec{\nabla} u = -\frac{\partial u}{\partial x}\hat{i} - \frac{\partial u}{\partial y}\hat{j} - \frac{\partial u}{\partial z}\hat{k}$$

$$-F_x = \frac{\partial}{\partial x}(ax^3 + dxy + exz) = 3ax^2 + dy + ez$$

$$-F_y = \frac{\partial}{\partial y}(by^3 + dxy + fyz) = 3by^2 + dx + fz$$

$$-F_z = \frac{\partial}{\partial z}(cz^3 + exz + fyz) = 3cz^2 + ex + fy$$

$$-F_x(x=1, y=2, z=3) = 3a + 2d + 3e$$

$$-F_y(1,2,3) = 3b2^2 + d + 3f = 12b + d + 3f$$

$$-F_z(1,2,3) = 3c3^2 + e + 2f = 27c + e + 2f$$

Let $a = b = c = 1$ dyne cm^{-2} and $d = e = f = 1$ dyne cm^{-1}.

Then find, at $P = (1,2,3)$

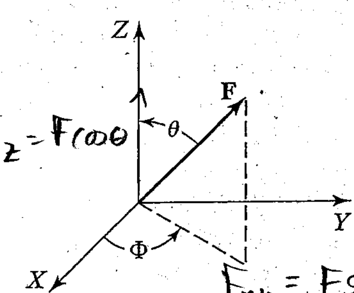
$$-F_x(P) = 3 + 2 + 3 = 8 \text{ dynes}$$

$$-F_y(P) = 12 + 1 + 3 = 16 \text{ dynes}$$

$$-F_z(P) = 27 + 1 + 2 = 30 \text{ dynes}$$

11,2

(B) Consider a vector **F** as shown in the following diagram:



$F_z = F\cos\theta$

$F_{xy} = F\sin\theta$

The direction of **F** may be specified by its spherical polar angles θ and ϕ, where θ is the angle between **F** and the z-axis and ϕ is the angle between the projection of **F** into the $(x$–$y)$ plane and the x-axis. If **F** is the force vector on a particle at the point given in (A), calculate the direction of **F** by computing the spherical polar angles θ and ϕ.

Use $F_x(P)$, $F_y(P)$, $F_z(P)$ from Part (A)

$$F^2 = F_x^2 + F_y^2 + F_z^2$$

$$F = \left[(-8)^2 + (-16)^2 + (-30)^2\right]^{1/2} = 35 \text{ dynes}$$

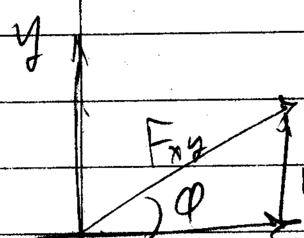
$$F_z = F\cos\theta \rightarrow \cos\theta = \frac{F_z}{F} = \frac{-30}{35} = -0.859 \text{ dynes}$$

$$\theta = 149^\circ$$

In the (x-y) plane, $F_{xy} = F\sin\theta = 35\sin(149^\circ)$

$F_{xy} = 17.9$ dynes

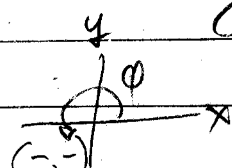
(not really necessary)



$F_y = F_{xy}\sin\varphi$

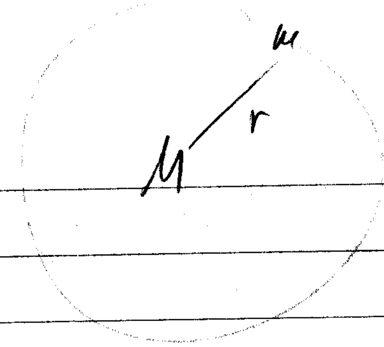
$F_x = F_{xy}\cos\varphi$

Find φ: $\dfrac{\sin\varphi}{\cos\varphi} = \tan\varphi = \dfrac{F_y/F_{xy}}{F_x/F_{xy}} = \dfrac{F_y}{F_x} = \dfrac{-16}{-8} = 2$



$\varphi \approx 243^\circ$

11.4 Assume that the hydrogen atom behaves like a planetary system, with the electron rotating about an extremely heavy nucleus. The mass of the electron is $m =$ 9.1094×10^{-31} kg. The kinetic energy of the electron $T =$ is 2.1792×10^{-18} J. If the nuclear mass is assumed to be infinite and the orbit circular with a radius of $r =$ 0.519×10^{-10} m, compute the angular momentum of the electron.



$$(11.22) \quad T = \frac{p_R^2}{2m} + \frac{p_\theta^2}{2mR^2}$$

Why is $p_R = 0$? $p_R = m\frac{dR}{dt}$ and $R =$ CONSTANT.

$$p_\theta = \sqrt{2mR^2T} =$$

$$= \sqrt{2 \times 9.11 \times 10^{-31}\,\text{kg} \times (0.52 \times 10^{-10}\,\text{m})^2 \left(2.18 \times 10^{-18}\,\text{kg}\frac{\text{m}^2}{\text{s}^2}\right)}$$

$$p_\theta = 1.03 \times 10^{-34}\,\frac{\text{kg m}^2}{\text{s}}$$

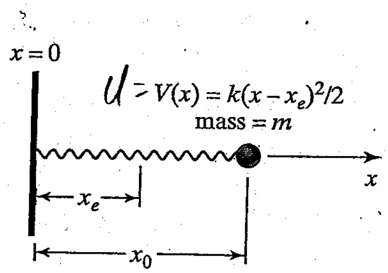
UNITS:

(Energy = Force · distance

Joules = Newton · m = $\frac{\text{kg m}}{\text{s}^2} \cdot \text{m} = \frac{\text{kg m}^2}{\text{s}^2}$)

11.6

Consider a single particle of mass m attached to a wall of infinite mass by a spring as shown in the following diagram:



The particle moves in one dimension (x). The potential for the system is $V(x) = k(x - x_e)^2/2$, where x_e is a constant. At time $t = 0$, the particle is at position x_o and its velocity is zero.

(A) Set up and solve the Newtonian equations of motion for this system. Plot the position of the particle as a function of time from $t = 0$ to $t = 4\pi(m/k)^{1/2}$. Make the abscissa of the plot be in units of $4\pi(m/k)^{1/2}$ so that the abscissa will go from 0 to 1. (*Hint:* Refer to Problem 9.5 for assistance in solving the differential equation.)

(B) As can be seen from the plot made in (A), the classical motion of the spring is oscillatory. In fact, this system is called a harmonic oscillator. The period τ of the oscillator is the time required to execute one complete vibrational cycle. The vibrational frequency ν is the reciprocal of the period. Determine the vibrational period and frequency of this harmonic oscillator in terms of k and m.

$$H = T + U = \frac{p_x^2}{2m} + \frac{1}{2}k(x - x_e)^2$$

✓ $$\frac{\partial H}{\partial p} = \frac{\partial x}{\partial t} = \frac{\partial}{\partial p_x}\left(\frac{p_x^2}{2m}\right) = \frac{p_x}{m} = V_x$$

$$\frac{\partial H}{\partial x} = -\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}k(x - x_e)^2\right)$$ let $z = x - x_e$, $dz = dx$

$$= \frac{\partial}{\partial z}\left(\frac{1}{2}k z^2\right) = kz$$

$$\frac{\partial U}{\partial x} = k(x - x_e)$$

EQN (11.6A)

$$ma = F$$

$$m\frac{d^2x}{dt^2} = -\frac{\partial U}{\partial x} = -k(x - x_e)$$

$$\frac{m\,d^2z}{dt^2} = -kz$$

$$\frac{d^2z}{dt^2} = -\frac{k}{m}z = -\omega^2 z$$

$$\omega = \sqrt{\frac{k}{m}}$$

Solve by inspection, as in class

Guess $z = z_0\cos(\omega t + \phi)$

$$\frac{dz}{dt} = -\omega z_0 \sin(\omega t + \phi)$$

$$\frac{d^2z}{dt^2} = -\omega^2 z_0 \cos(\omega t + \phi)$$

$$= -\omega^2 z$$

11.6 cont'd: $z = z_0 \cos(\omega t + \phi)$ where $\omega = \sqrt{\frac{k}{m}}$

$$(x - x_e) = (x_0 - x_e) \cos(\omega t + \phi)$$

Apply initial conditions: $x(t=0) = x_0$

$$z(t=0) = z_0 = x_0 - x_e$$

$$\frac{dx}{dt} = \frac{dz}{dt} = v(t=0) = 0$$

at $t=0$, $\frac{dz}{dt} = -\omega z_0 \sin(\omega t + \phi) = 0$

$$\sin(0 + \phi) = 0$$

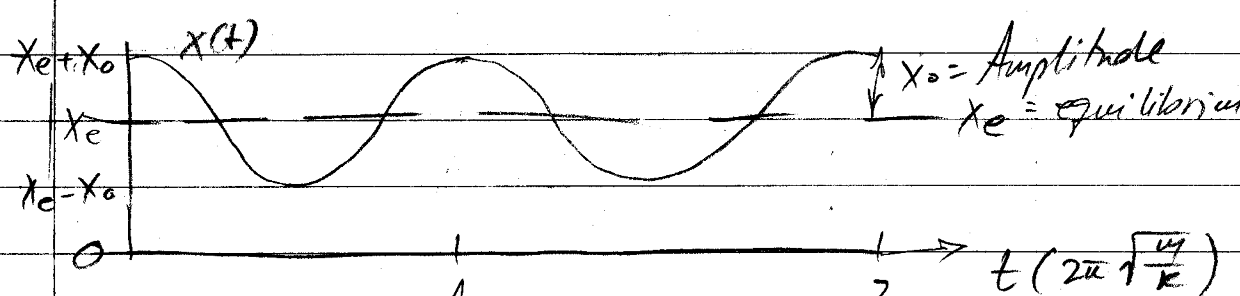
$$\sin \phi = 0$$

$$\phi = 0 \quad \text{(no phase shift)}$$

$$x = x_e + (x_0 - x_e)\cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$v = -\sqrt{\frac{k}{m}}(x_0 - x_e)\sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} \rightarrow$ PERIOD $T = 2\pi\sqrt{\frac{m}{k}}$ so $2T = 4\pi\sqrt{\frac{m}{k}}$



We already showed that the period $T = 2\pi\sqrt{\frac{m}{k}}$

angular frequency $\omega = \sqrt{\frac{k}{m}}$ (rad/s)

vibrational frequency $f = \nu = \frac{\omega}{2\pi} = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

11.5

Assume the particle illustrated in Figure 11.5 is moving in three-dimensional space. Then, in rectangular Cartesian coordinates, the kinetic energy of the particle is

$$T = \frac{m}{2}[v_x^2 + v_y^2 + v_z^2].$$

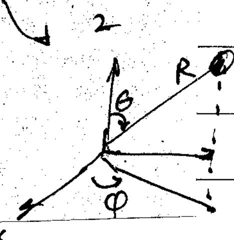
Suppose we wish to transform to a spherical polar coordinate system with coordinates (R, θ, ϕ) instead of (x, y, z). This coordinate system is illustrated in the figure given in Problem 11.2 if we replace F with R. The transformation equations between the spherical polar coordinate system and a rectangular Cartesian system, given in Chapter 9 (Figure 9.3) are

$$x = R \sin\theta \cos\phi,$$

$$y = R \sin\theta \sin\phi,$$

and

$$z = R \cos\theta.$$

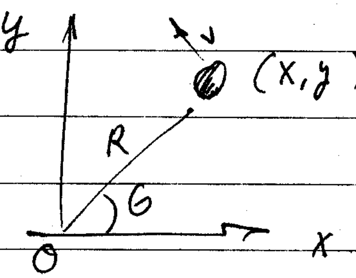


(A) Obtain the kinetic energy of the particle in terms of the spherical polar coordinates and their rates of change with time.

(B) Obtain expressions for the momenta conjugate to the spherical polar coordinates.

(C) Show that the momentum conjugate to ϕ, P_ϕ, is the same as the Cartesian z-component of angular momentum.

(D) Express the kinetic energy in terms of the spherical polar conjugate momenta and coordinates.



(Fig 11.5)

(A) $V_x = \frac{dx}{dt} = \frac{dR}{dt}\sin\theta\cos\phi + R\cos\phi\frac{d}{dt}\sin\theta + R\sin\theta\frac{d}{dt}\cos\phi$

$= \underbrace{\frac{dR}{dt}}_{V_r}\sin\theta\cos\phi + R\cos\phi\cos\theta\underbrace{\frac{d\theta}{dt}}_{\omega_\theta} + R\sin\theta(-\sin\phi)\underbrace{\frac{d\phi}{dt}}_{\omega_\phi}$

$V_y = \frac{dy}{dt} = \frac{dR}{dt}\sin\theta\sin\phi + R\sin\phi\frac{d}{dt}\sin\theta + R\sin\theta\frac{d}{dt}\sin\phi$

$= V_r\sin\theta\sin\phi + R\sin\phi\cos\theta\,\omega_\theta + R\sin\theta\cos\phi\,\omega_\phi$

$V_z = \frac{dz}{dt} = \frac{dR}{dt}\cos\theta + R\frac{d}{dt}\cos\theta$

$= V_r\cos\theta - R\omega_\theta\sin\theta$

$$V_x^2 = (V_r \sin\Theta \cos\Phi + R\omega_\theta \cos\Phi \cos\Theta - R\omega_\varphi \sin\Theta \sin\Phi)^2 = (a+b-c)(a+b-c) = a^2+b^2+c^2+2ab-2ac-2bc$$

$$= V_r^2 \sin^2\Theta \cos^2\Phi + (R\omega_\theta)^2 \cos^2\Phi \cos^2\Theta + (R\omega_\varphi)^2 \sin^2\Theta \sin^2\Phi + 2V_r R\omega_\theta \sin\Theta \cos\Phi \cos\Phi \cos\Theta$$

$$- 2V_r \cancel{R}\omega_\varphi \sin\Theta \cos\Phi \sin\Theta \sin\Phi - 2R\omega_\theta \cancel{R}\omega_\varphi \cos\Phi \cos\Theta \sin\Theta \sin\Phi$$

$$V_y^2 = (V_r \sin\Theta \sin\Phi + R\omega_\theta \sin\Phi \cos\Theta + R\omega_\varphi \sin\Theta \cos\Phi)^2$$

$$+ 2R\cancel{\omega_\theta} R\omega_\varphi \sin\Phi \cos\Theta \sin\Theta \cos\Phi$$

$$= V_r^2 \sin^2\Theta \sin^2\Phi + (R\omega_\theta)^2 \sin^2\Phi \cos^2\Theta + (R\omega_\varphi)^2 \sin^2\Theta \cos^2\Phi + 2V_r R\omega_\theta \sin\Theta \sin^2\Phi \cos\Theta + 2V_r \cancel{R}\omega_\varphi \sin^2\Theta \sin\Phi \cos\Phi$$

$$V_z^2 = (V_r \cos\Theta - R\omega_\theta \sin\Theta)^2 = V_r^2 \cos^2\Theta + (R\omega_\theta)^2 \sin^2\Theta - 2V_r R\omega_\theta \cos\Theta \sin\Theta$$

$$V_x^2 + V_y^2 = V_r^2 \sin^2\Theta(\cos^2\Phi + \sin^2\Phi) + (R\omega_\theta)^2 \cos^2\Theta(\cos^2\Phi + \sin^2\Phi) + (R\omega_\varphi)^2 \sin^2\Theta(\sin^2\Phi + \cos^2\Phi) + 2V_r R\omega_\theta \sin\Theta\cos\Theta(\cos^2\Phi + \sin^2\Phi)$$

$$= V_r^2 \sin^2\Theta \qquad + (R\omega_\theta)^2 \cos^2\Theta \qquad + (R\omega_\varphi)^2 \sin^2\Theta \qquad + 2V_r R\omega_\theta \sin\Theta\cos\Theta$$

$$V_x^2 + V_y^2 + V_z^2 = V_r^2(\sin^2\Theta + \cos^2\Theta) + (R\omega_\theta)^2(\sin^2\Theta + \cos^2\Theta) + 2V_r R\omega_\theta(-\cos\Theta\sin\Theta + \sin\Theta\cos\Theta) + (R\omega_\varphi)^2 \sin^2\Theta$$

$$= V_r^2 \qquad + (R\omega_\theta)^2 \qquad + 0 \qquad + (R\omega_\varphi)^2 \sin^2\Theta$$

$$\underline{V_x^2 + V_y^2 + V_z^2 = V_r^2 + (R\omega_\theta)^2 + (R\omega_\varphi \sin\Theta)^2}$$

$$T = \frac{1}{2}mV^2 = \frac{m}{2}\left(V_r^2 + (R\omega_\theta)^2 + (R\omega_\varphi \sin\Theta)^2\right)$$

In Raff's notation $V_\theta = \omega_\theta$, $V_\varphi = \omega_\varphi$. Note these are ANGULAR velocities,

So I will use our usual physics notation:

$$\omega_\theta = \frac{d\theta}{dt} \text{ and } \omega_\varphi = \frac{d\varphi}{dt}.$$

Group similar terms, then simplify using $\sin^2 + \cos^2 = 1$

$$V_x^2 + V_y^2 + V_z^2 = V_r^2 \sin^2\theta \cos^2\varphi + V_r^2 \cos^2\theta + V_r^2 \sin^2\theta \sin^2\varphi$$

$$(V_r^2 \sin^2\theta + V_r^2 \cos^2\theta = V_r^2)$$

$$+(R\omega_\theta)^2 \left[\cos^2\varphi \cos^2\theta + \sin^2\theta + \sin^2\varphi \cos^2\theta = \cos^2\theta(\cos^2\varphi + \sin^2\varphi = 1) + \sin^2\theta \right.$$
$$\left. = \cos^2\theta + \sin^2\theta = 1 \right]$$

$$+(R\omega_\varphi)^2 \left[\sin^2\theta \sin^2\varphi + \sin^2\theta \cos^2\varphi = \sin^2\theta(\sin^2\varphi + \cos^2\varphi) = \sin^2\theta\right]$$

$$+V_r\left[\sin\theta\, R\omega_\theta\{\cos\varphi(\cos\varphi\cos\theta) + \sin\varphi(\sin\varphi\cos\theta)\} - R\omega_\theta \cos\theta \sin\theta\right]$$
$$= V_r R\omega_\theta\left[\sin\theta\{(\cos^2\varphi + \sin^2\varphi)\cos\theta = \cos\theta\} - \cos\theta\sin\theta\right]$$
$$[\sin\theta\cos\theta - \cos\theta\sin\theta] = 0$$
$$+V_r \sin\theta\left[R\omega_\varphi\{\cos\varphi(-\sin\theta\sin\varphi) + \sin\varphi(\sin\theta\cos\varphi) = 0\right]$$

$$V_x^2 + V_y^2 + V_z^2 = V_r^2 + (R\omega_\theta)^2 + (R\omega_\varphi)^2 \sin^2\theta$$

$$T = \frac{1}{2} m V^2 = \frac{m}{2}\left(V_r^2 + (R\omega_\theta)^2 + (R\omega_\varphi \sin\theta)^2\right)$$

In Raff's notation $\omega_\theta = V_\theta$ and $\omega_\varphi = V_\varphi$

But note that these are ANGULAR velocities
So I will use our usual physics notation
hereafter: $\omega_\theta = \frac{d\theta}{dt}$ and $\omega_\varphi = \frac{d\varphi}{dt}$

We found: $T = \frac{m}{2}\left(V_r^2 + (RW_\theta)^2 + (RW_\varphi \sin\theta)^2\right)$

(B) MOMENTA conjugate to (r, θ, ϕ) are $(p_r, p_\theta, p_\varphi)$:

$$p_r = \frac{\partial T}{\partial V_r} = \frac{m}{2}\frac{\partial}{\partial V_r}(V_r^2) = \frac{m}{2}\,2V_r = \underline{mV_r = p_r} \quad \text{LINEAR}$$

$$p_\theta = \frac{\partial T}{\partial W_\theta} = \frac{m}{2}\frac{\partial}{\partial w_\theta}(Rw_\theta)^2 = \underline{mR^2 w_\theta} \quad \text{ANGULAR } (L_\theta)$$

$$p_\phi = \frac{\partial T}{\partial W_\phi} = \frac{m}{2}\frac{\partial}{\partial w_\varphi}(Rw_\varphi \sin\theta)^2 = mR^2 w_\varphi \sin^2\theta \quad \text{ANGULAR } (L_\phi)$$

This is the angular momentum about z-axis: L_z or M_z

(C) Angular momentum about z axis is $L_z = M_z = p_\varphi$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{M} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ V_x & V_y & V_z \end{vmatrix} = \begin{matrix} \hat{i}\,(yV_z - zV_y) \\ -\hat{j}\,(xV_z - zV_x) \\ +\hat{k}\,(xV_y - yV_x) \end{matrix}$$

$$L_z = M_z = (xV_y - yV_x)$$
$$= R\sin\theta\cos\phi\,(V_y) - R\sin\theta\sin\phi\,(V_x)$$

Insert V_y & V_x: $= R\sin\theta\cos\phi(V_r\sin\theta\sin\phi + Rw_\theta\sin\varphi\cos\theta + Rw_\varphi\sin\theta\cos\phi)$

Simplify: $- R\sin\theta\sin\phi(V_r\sin\theta\cos\phi + Rw_\theta\cos\varphi\cos\theta - Rw_\varphi\sin\theta\sin\varphi)$

$$= V_r R\sin\theta\,[\cos\phi\sin\theta\sin\varphi - \sin\phi\sin\theta\cos\varphi = 0]$$
$$+ R^2 w_\theta\sin\theta\,[\cos\varphi\sin\varphi\cos\theta - \sin\varphi\cos\varphi\cos\theta = 0]$$
$$+ R^2 w_\varphi\sin\theta\,[\cos\phi\sin\theta\cos\phi + \sin\phi(\sin\theta\sin\varphi)]$$
$$[\cos^2\varphi\sin\theta + \sin^2\varphi\sin\theta = \sin\theta]$$

Compare to p_φ: $L_z = R^2 w_\varphi \sin^2\theta = p_\varphi$ ✓

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We found: $P_\Theta = mR^2\omega_\theta$ $\quad P_\varphi = mR^2\omega_\varphi \sin^2\Theta$

(1) Express T in terms of (R, Θ, φ) and $(p_r, P_\Theta, P_\varphi)$

$$\frac{1}{2} m V_R^2 = \frac{P_R^2}{2m}, \quad \frac{1}{2} m (R\omega_\theta)^2 = \frac{m}{2} R^2 \omega_\theta^2 = \frac{(mR^2\omega_\theta)^2}{2mR^2} = \frac{P_\Theta^2}{2mR^2}$$

$$\frac{1}{2} m (R\omega_\varphi \sin\Theta)^2 = \frac{m}{2} R^2 \omega_\varphi^2 \sin^2\Theta = \frac{(mR^2\omega_\varphi \sin^2\Theta)^2}{2mR^2\sin^2\Theta} = \frac{P_\varphi^2}{2mR^2\sin^2\Theta}$$

$$T = \frac{1}{2} m \left[V_R^2 + (RV_\Theta)^2 + (RV_\varphi \sin\Theta)^2 \right]$$

$$T = \frac{P_R^2}{2m} + \frac{P_\Theta^2}{2mR^2} + \frac{P_\varphi^2}{2mR^2\sin^2\Theta}$$

We have shown that this Kinetic energy in spherical coordinates is the same as that in Cartesian coordinates:

$$T = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m}$$

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