ZITA
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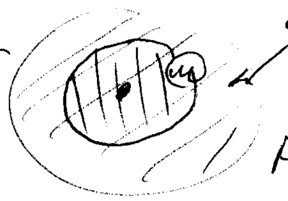
Derive the shell theorem – following Carroll & Ostlie Ch. 2

- The gravitational force due to a spherically symmetric mass M is just the same as that due to a POINT mass M at its center (acting on objects outside the mass distribution) $\therefore \boxed{F = \frac{GmM}{r^2}}$
- Corrolary: the force on a test mass (m) INSIDE the mass distribution is just the same as that due to ONLY the mass inside the (m) radius, concentrated at the center



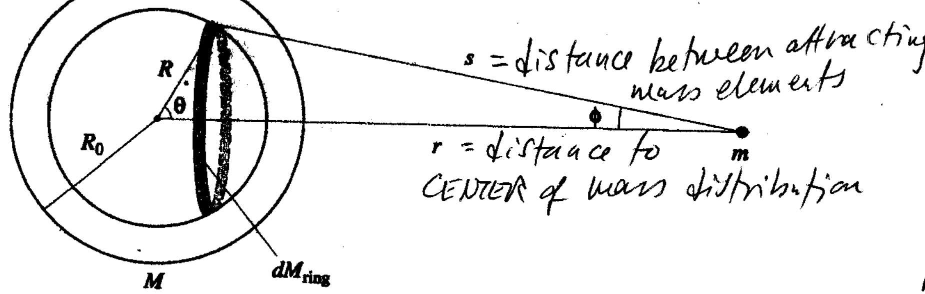
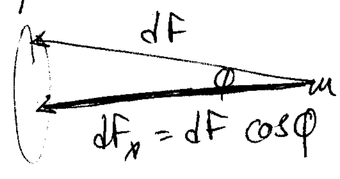


Figure 2.8 The gravitational effect of a spherically symmetric mass distribution.

$$dF = \frac{Gm\,dM}{s^2}$$

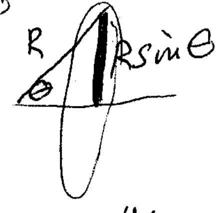
but since the $\updownarrow$ components of dF due to pairs of mass elements on the ring CANCEL, only the $\leftarrow$ components survive.

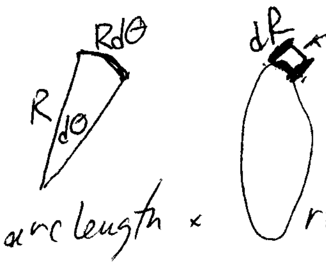


If dM_{ring} is the mass of the *ring* being considered, the force exerted by that ring on m is given by

$$dF_{ring} = G\frac{m\,dM_{ring}}{s^2}\cos\phi.$$

The volume of the ring is its cross-sectional area dA × its circumference $2\pi R\sin\theta$





$dA = R\,d\theta\,dR$

arc length × ring thickness $= dA$

Volume $dV_{ring} = (dA)\,2\pi R\sin\theta$
$= (R\,d\theta\,dR)\,2\pi R\sin\theta$

Assuming that the mass density, $\rho(R)$, of the extended object is a function of radius only and that the volume of the ring of thickness dR is dV_{ring}, then

density $= \frac{\text{mass}}{\text{volume}}$

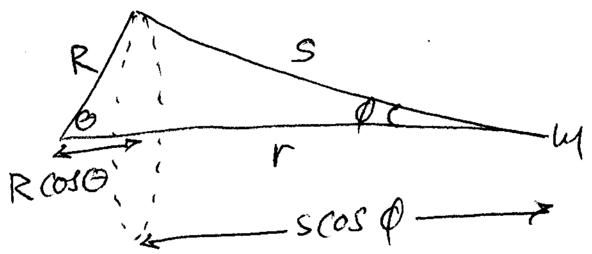
$\rho = \frac{dM}{dV}$

$$\begin{aligned} dM_{ring} &= \rho(R)\,dV_{ring} \\ &= \rho(R)2\pi R\sin\theta R\,d\theta\,dR \\ &= 2\pi R^2\rho(R)\sin\theta\,dR\,d\theta. \end{aligned}$$

So far, we have the force element due to a ring of mass dM on a test mass m:

$$dF_{ring} = \frac{Gm(dM_{ring})}{S^2}\cos\phi$$

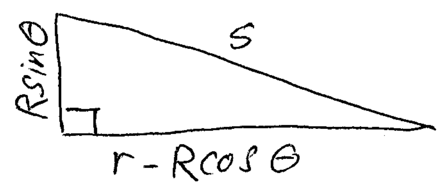
$$= \frac{Gm}{S^2}\left[\cos\phi\right]\left(2\pi R^2 \rho \sin\theta \, dR \, d\theta\right)$$



By geometry, $r = R\cos\theta + s\cos\phi$, so

$$\left[\cos\phi = \frac{r - R\cos\theta}{s}\right]$$

where s may be found by the Pythagorean theorem,



$$s = \sqrt{(r - R\cos\theta)^2 + R^2\sin^2\theta}$$

$$S = \sqrt{r^2 - 2rR\cos\theta + R^2}. = (r^2 - 2rR\cos\theta + R^2)^{1/2}$$

$$S^3 = (r^2 - 2rR\cos\theta + R^2)^{3/2}$$

Substituting into the expression for dF_{ring}, summing over all *rings* located at a distance R from the center of the mass M (i.e., integrating over all θ from 0 to π for constant R) and then summing over all resultant *shells* of radius R from $R = 0$ to $R = R_0$ gives the total force of gravity acting on the small mass

$$dF_{ring} = \frac{Gm}{S^2}\left[\frac{r - R\cos\theta}{S}\right] 2\pi R^2 \rho \sin\theta \, dR \, d\theta$$

$$= 2\pi Gm \frac{(r - R\cos\theta)}{S^3} R^2 \rho \sin\theta \, dR \, d\theta$$

$$= 2\pi Gm \frac{(r - R\cos\theta)}{(r^2 - 2rR\cos\theta + R^2)^{3/2}} R^2 \rho \sin\theta \, dR \, d\theta$$

$$F = Gm\int_0^{R_0}\int_0^{\pi} \frac{(r - R\cos\theta)\rho(R)2\pi R^2 \sin\theta}{s^3}\, d\theta\, dR \quad = Gm\iint \frac{(r - R\cos\theta)\rho 2\pi R^2}{u^{3/2}}\left(\frac{du}{2rR}\right) dR$$

$$= 2\pi Gm \int_0^{R_0}\int_0^{\pi} \frac{rR^2\rho(R)\sin\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}}\, d\theta\, dR \quad - 2\pi Gm \int_0^{R_0}\int_0^{\pi} \frac{R^3\rho(R)\sin\theta\cos\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}}\, d\theta\, dR.$$

$$F = \underbrace{2\pi Gm \iint \frac{rR^2\rho}{u^{3/2}}\left(\frac{du}{2rR}\right) dR}_{= F_1} \quad \underbrace{- 2\pi Gm \iint \frac{R^3\rho\cos\theta}{u^{3/2}}\left(\frac{du}{2rR}\right) dR}_{- F_2}$$

The integrations over θ may be carried out by making the change of variable, $u \equiv s^2 = r^2 + R^2 - 2rR\cos\theta$. Then $\cos\theta = (r^2 + R^2 - u)/2rR$ and $\sin\theta \, d\theta = du/2rR$.

$$\frac{du}{d\theta} = \frac{d}{d\theta}(r^2 + R^2) - 2rR\frac{d}{d\theta}\cos\theta$$

$$\frac{du}{d\theta} = 0 - 2rR(-\sin\theta)$$

$$\leftarrow du = 2rR\sin\theta \, d\theta$$

$F = F_1 - F_2$ where $F_1 = \pi G m \iint \rho R \frac{du}{u^{3/2}} dR$

and $F_2 = \pi G m \iint \rho \frac{R^2}{r}(\cos\theta) \frac{du}{u^{3/2}} dR$

Let's integrate F_1 first.

$$\int \frac{du}{u^{3/2}} = \int u^{-3/2} du = \frac{u^{-1/2}}{-1/2} = \frac{-2}{\sqrt{u}}\Bigg|_{\theta=0}^{\theta=\pi} = -2\left[\frac{1}{\sqrt{r^2+R^2-2rR\cos\pi}} - \frac{1}{\sqrt{r^2+R^2-2rR\cos 0}}\right]$$

$\cos\pi = -1$ $\quad$ $\cos 0 = 1$

$$= +2\left[\frac{-1}{\sqrt{r^2+R^2+2rR}} + \frac{1}{\sqrt{r^2+R^2-2rR}}\right] = 2\left[\frac{-1}{\sqrt{(r+R)^2}} + \frac{1}{\sqrt{(r-R)^2}}\right]$$

$$\int u^{-\frac{3}{2}} du = 2\left[\frac{1}{r-R} - \frac{1}{r+R}\right] = 2\left[\frac{r+R-(r-R)}{(r+R)(r-R)}\right] = \frac{4R}{(r^2-R^2)}$$

Then $F_1 = \pi G m \int \rho R \left(\frac{4R}{r^2-R^2}\right) dR = \underline{4\pi G m \int \rho \frac{R^2}{r^2-R^2} dR}$

Next, $F_2 = \pi G m \iint \rho \frac{R^2}{r}\left(\frac{r^2+R^2-u}{2rR}\right)\frac{du}{u^{3/2}} dR = \frac{\pi G M}{2r^2}\iint \rho R \frac{(r^2+R^2-u)}{u^{3/2}} du\, dR$

$F_2 = \frac{\pi G M}{2r^2}\int \rho R (I_1 - I_2) dR$ where $I_1 = \int \frac{r^2+R^2}{u^{3/2}} du$ and $I_2 = \int \frac{u\, du}{u^{3/2}}$

$$I_1 = (r^2+R^2)\int u^{-3/2} du = (r^2+R^2)\frac{4R}{(r^2-R^2)} \quad \text{(done above)}$$

$$I_2 = \int \frac{u\,du}{u^{3/2}} = \int u^{-1/2} du = \frac{u^{1/2}}{1/2}\Bigg|_{\theta=0}^{\theta=\pi} = 2\left[\sqrt{r^2+R^2-2rR\cos\pi} - \sqrt{r^2+R^2-2rR\cos 0}\right]$$

$$= 2\left[(r-R)-(r+R)\right] = 2\left[-2R\right] = -4R$$

(ii) $I_1 - I_2 = 4R\left[\frac{(r^2+R^2)}{(r^2-R^2)} - 1\right] = 4R\left[\frac{(r^2+R^2)-(r^2-R^2)}{(r^2-R^2)}\right] = 4R\left[\frac{-2R^2}{(r^2-R^2)}\right]$

Then $F_2 = \frac{\pi G m}{2 r^2}\int \rho R\left(4R\left[\frac{-2R^2}{r^2-R^2}\right]\right) dR = \frac{4\pi G M}{2}\int \rho \frac{2R^4/r^2}{(r^2-R^2)}\, dR$

$$F_1 - F_2 = 4\pi G m \int \rho \frac{R^2}{r^2-R^2}\left(1-\frac{R^2}{r^2}\right) dR$$

$$\left(1-\frac{R^2}{r^2} = \frac{r^2-R^2}{r^2}\right)$$

$$F_1 - F_2 = 4\pi G m \int \rho \frac{R^2}{(r^2-R^2)}\,\frac{(r^2-R^2)}{r^2}\, dR$$

$$F = \frac{4\pi G M}{r^2}\int \rho R^2\, dR \quad \checkmark$$

After the appropriate substitutions and integration over the new variable u, the equation for the force becomes

$$F = \frac{Gm}{r^2}\int_0^{R_0} 4\pi R^2 \rho(R)\, dR.$$

Notice that the integrand is just the mass of a *shell* of thickness dR, having a volume dV_{shell}, or

$$\begin{aligned} dM_{\text{shell}} &= 4\pi R^2 \rho(R)\, dR \\ &= \rho(R)\, dV_{\text{shell}}. \end{aligned}$$

Therefore the integrand gives the force on m due to a spherically symmetric mass shell of mass dM_{shell} as

$$dF_{\text{shell}} = \frac{Gm\, dM_{\text{shell}}}{r^2}.$$

The shell acts gravitationally as if its mass were located entirely at its center. Finally, integrating over the mass shells, we have that the force exerted on m by an extended, spherically symmetric mass distribution is directed along the line of symmetry between the two objects and is given by

$$F = G\frac{Mm}{r^2},$$

just the equation for the force of gravity between two point masses.