Axiomatic Geometry

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Acknowledgements

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Introduction

This guide was designed for a sophmore/junior-level course in axiomatic geometry. It follows a two-semester sequence in the foundations of mathematics, and independent theorem-proving is the central focus of all three courses. It precedes real analysis and abstract algebra for mathematics majors, while elementary ed math students take it (often with trepidation!) as their final mathematics course. I have deliberately chosen a thorough, ground-up axiomatic approach, at the cost of not covering more advanced topics in geometry per se.

Chapter 1 presents several diverse models of incidence and betweenness geometry which play a central role throughout the course. I emphasize that a statement of geometry is only true or false in a model; that as we add axioms, we eliminate models, thereby narrowing the discourse to those models which reflect what most interests us. Ultimately, scientists would like to understand the geometry of the space-time that we live in, but it is not yet clear exactly which axioms it satisfies. The sphere model adds to the diversity in an important way. Several times, I tried to introduce the projective plane as pairs of antipodal points of the sphere, but found this incomprehensible to my audience, so I have replaced it with the sphere in which the first incidence axiom fails.

Theorem-proving begins in Chapter 2 with the incidence and betweenness axioms. Quickly we see how models can guide our proofs: Pasch's Theorem is clearly false in $\mathbb{R}^3$; therefore any proof of this theorem would necessarily utilize some axiom that fails in $\mathbb{R}^3$. Together with the congruence axioms in Chapter 3, these two chapters develop a rather conventional sequence of ideas in neutral geometry. Along the way, I present the "bumpy planes" which are incidence-betweenness isomorphic to $\mathbb{R}^2$ and satisfy all of the congruence axioms except SAS. This shows how SAS is really saying that the curvature of the space is uniform, which significantly limits the possible models. Here I like to mention the Hyperbolic Plane, positive and negative curvature, and how they relate to the parallel axioms and to current developments in cosmology and the structure of the universe. I do this judiciously, when no-one has something to present and interest seems to be sufficient. Without going deeply into the subject, I find that by the end of Chapter 3, students have a solid working sense of how geometry is developed. With this infrastructure in place, one could easily add further theorems for a class that is able to handle more material.

I offer these notes as a starting point for the instructor who would like to teach a discovery-based course in axiomatic geometry. This guide has evolved over many iterations, and now works well for most of the mathematics students at SUNY New Paltz, a moderately competitive 4-year state college. Depending on your audience, you may well need to modify them, either by adding additional material and removing unnecessary intermediate lemmas, or by deleting material and adding further intermediate lemmas. Regardless of your audience, you may choose to replace some parts of this guide with topics, models, or theorems that match your personal interests. I would certainly appreciate any corrections, comments, or ideas that you care to share.

David Clark August 2002

1 Models of Geometry

A *model* $\mathbb{G}$ *of geometry* consists of a set of objects which we call *points*, a collection of special subsets of those objects that we call *lines*, and a *betweenness* relation among the points. We write "$A \in m$" to mean that point A is in (on) the line m, and we write "$[A, B, C]$" to indicate that the point B is between the points A and C. To specify a model $\mathbb{G}$, then, we must

- specify the set of points of $\mathbb{G}$;
- tell which subsets of the points are the lines of $\mathbb{G}$; and then
- tell which points are between which other points.

Here are some examples.

$\mathbb{R}^2$ (Euclidean 2-Space)

Our first example is the model that is most familiar. A *point* is an ordered pair (x, y) of real numbers. Points of $\mathbb{R}^2$ can be added and multiplied by real numbers using the definitions

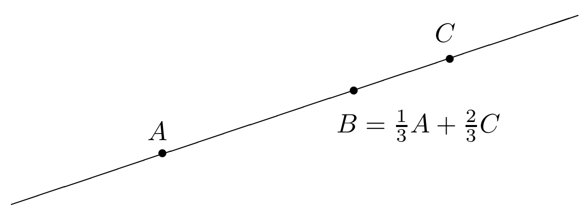
$$(a, b) + (c, d) = (a + c, b + d)$$
$$t(a, b) = (ta, tb)$$

If A and D are points of $\mathbb{R}^2$ where $D \neq (0, 0)$, then the set

$$m := \{X \mid X = tD + A\}$$

is a *line* of $\mathbb{R}^2$. If A and C are distinct points of $\mathbb{R}^2$, then a point B is *between* A and C if there is a number t between 0 and 1 such that

$$B = A + t(C - A) = (1 - t)A + tC$$

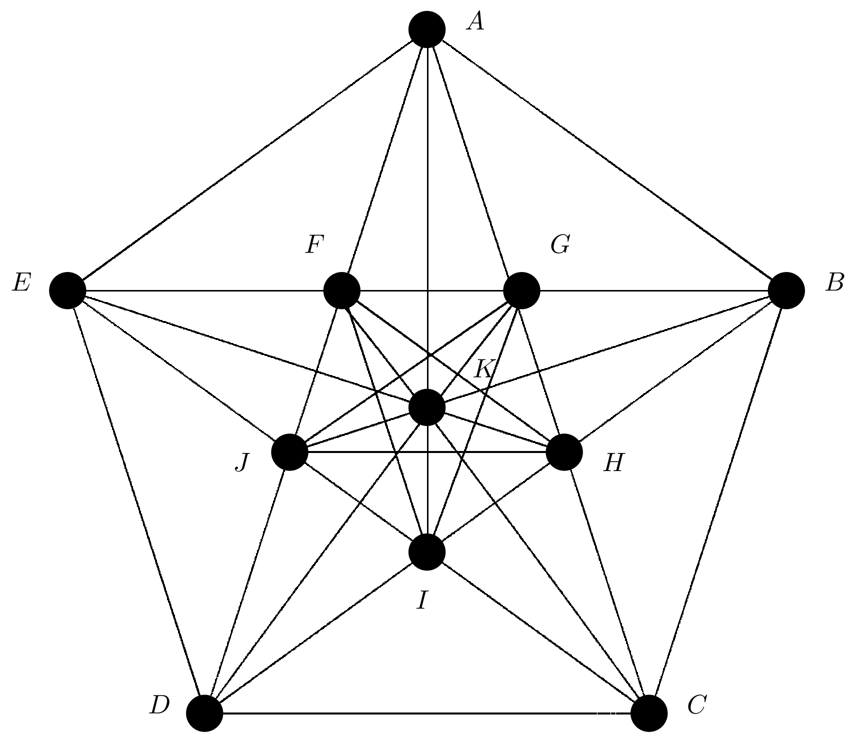


$\mathbb{D}$ (the Disc Model)

A *point* of $\mathbb{D}$ is a point (x, y) of $\mathbb{R}^2$ where $x^2 + y^2 < 1$. The intersection of a line of $\mathbb{R}^2$ with the points of $\mathbb{D}$ is a *line* of $\mathbb{D}$ provided that it is not empty. Given points A, B, and C of $\mathbb{D}$, we say that B is *between* A and C if $[A, B, C]$ is true in $\mathbb{R}^2$.

$\mathbb{P}$ (the Pentagon Model)

This rather different model, which has only 11 *points* and 20 *lines*, is illustrated in the diagram below. (A set of points is a line if it is the set of all points on one of the 20 straight line segments drawn.) A point Y is *between* points X and Z if X, Y, and Z are distinct points on one of the 20 lines, and Y appears between X and Z in the diagram.



$\mathbb{Q}^2$ (the Rational Plane)

A *point* of $\mathbb{Q}^2$ is a point of $\mathbb{R}^2$ whose coordinates are both rational numbers. We would like to define a line of $\mathbb{Q}^2$ to be the intersection $m^Q := m \cap Q^2$ of a line m of $\mathbb{R}^2$ with the set Q^2 of points of $\mathbb{Q}^2$. But if we took m to be the line whose equation is $y = \sqrt{2}$, then m^Q would be the empty set.

In order to avoid degenerate lines, we define a *rational line* to be a line of R^2 that contains at least two rational points. A *line of* $\mathbb{Q}^2$ is a set m^Q of points of $\mathbb{Q}^2$, where m is a rational line of R^2. For rational points A, B, and C, we say that B is *between* A and C if $[A, B, C]$ is true in $\mathbb{R}^2$.

$\mathbb{R}^3$ (Euclidean 3-Space)

As a model of geometry, three-dimensional Euclidean space is very similar to two-dimensional Euclidean space. A *point* is an ordered triple (x, y, z) of real numbers. Points of $\mathbb{R}^3$ can be added and multiplied by real numbers using the definitions

$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$$
$$t(a, b, c) = (ta, tb, tc)$$

If A and D are points of $\mathbb{R}^3$ where $D \neq (0, 0, 0)$, then the set

$$m := \{X \mid X = tD + A\}$$

is a *line* of $\mathbb{R}^3$. If A and C are distinct points of $\mathbb{R}^3$, then a point B is *between* A and C if there is a number t between 0 and 1 such that

$$B = A + t(C - A) = (1 - t)A + tC$$

$\mathbb{S}$ (The Sphere)

Consider a sphere in $\mathbb{R}^3$ with center $(0, 0, 0)$ and radius 1. A *point* is a point (x, y, z) of $\mathbb{R}^3$ satisfying the equation $x^2 + y^2 + z^2 = 1$. A *line* is a circle with center $(0, 0, 0)$ and radius 1 (called a *great circle* of the sphere). Point B is *between* points A and C if A and C are not opposite points ($A \neq -C$), and B lies along the shorter great circle path from A to C. Betweenness is only defined for points that are reasonably close together.

Later, we will include additional models of geometry. Notice that the terms "point", "line", and "between" do not have any fixed meaning in the context of geometry. Rather, they are only defined in a particular model of geometry. For this reason, they are normally referred to, perhaps somewhat misleadingly, as *undefined terms*. In contrast, *defined terms* are terms that are defined generally, for all models at once, using the undefined terms. Here are a few examples of defined terms. Given distinct points A and B,

- the *segment* AB is the set of points consisting of A, B, and all points between A and B;
- the *ray* $\overrightarrow{AB}$ is the union of the segment AB and the set of points X such that $[A, B, X]$:

$$\overrightarrow{AB} := \{X \mid X = A,\ X = B,\ [A, X, B] \text{ or } [A, B, X]\}$$

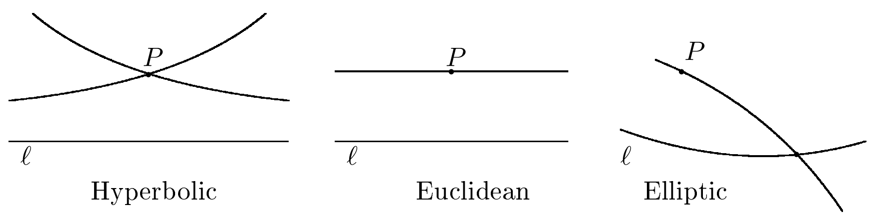
Lines ℓ and m are said to be *parallel* if they have no point in common; that is, $\ell \cap m = \emptyset$. A set of points is *collinear* if some line contains them all.

The issue of the existence and uniqueness of parallel lines has played a central role in the history of geometry, and will be a theme we will follow in this course. It turns out that in most interesting models of geometry, parallel lines exist in one of the following three regular patterns. Notice that a model of geometry can satisfy at most one of these properties.

Hyperbolic Parallel Property: For every line ℓ and point $P \notin \ell$, there are at least two lines containing P that are parallel to ℓ.

Euclidean Parallel Property: For every line ℓ and point $P \notin \ell$, there is exactly one line containing P that is parallel to ℓ.

Elliptic Parallel Property: For every line ℓ and point $P \notin \ell$, there are no lines containing P that are parallel to ℓ; that is, there are no parallel lines at all.



We will be interested in knowing which, if any, of these parallel properties hold, in each of the geometries that we will study.

Problems

State and justify each answer. Either prove or disprove each statement.

1. $\mathbb{R}^2$ **(Euclidean 2-Space)**

   (i) The point $B = (16, 11)$ is between $A = (5, 7)$ and $C = (49, 23)$.

   (ii) The point $B = (15, 12)$ is between $A = (4, 8)$ and $C = (50, 22)$.

2. $\mathbb{D}$ **(the Disc Model)**

   (i) Does $\mathbb{D}$ have a line which contains exactly one point?

   (ii) How many lines are there containing the point $P = (0, 0.5)$ that are parallel to the line $\ell = \{(x, 0) \mid -1 < x < 1\}$?

3. $\mathbb{P}$ **(the Pentagon Model)**

   (i) List all lines that are parallel to the line $\ell = \{E, F, G, B\}$.

(ii) How many triples (X, Y, Z) of points are there where $[X, Y, Z]$?

(iii) List all of the points of the rays $\overrightarrow{CH}$ and $\overrightarrow{CG}$.

4. $\mathbb{Q}^2$ **(the Rational Plane)**

(i) Is there a line of $\mathbb{R}^2$ that contains exactly one rational point?

(ii) Is there a line of $\mathbb{R}^2$ that contains exactly two rational points?

(iii) Is it possible to have rational lines ℓ and m of $\mathbb{R}^2$, where ℓ and m are not parallel in $\mathbb{R}^2$, but ℓ^Q and m^Q are parallel in $\mathbb{Q}^2$?

5. $\mathbb{R}^3$ **(Euclidean 3-Space)**

The model $\mathbb{R}^2$ has the property that two lines parallel to the same line are parallel to each other. Show that this is not true in $\mathbb{R}^3$.

6. $\mathbb{S}^2$ **(The Sphere)**

Consider the 3 collinear points $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, -1, 0)$, lying on the line $\ell = \{(x, y, 0) \mid x^2 + y^2 = 1\}$. Make 4 sketches of ℓ to illustrate segments AB and AC and rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (Since ℓ lies entirely in the xy-plane, that is all you need to draw.)

7. Which of our models of geometry satisfy the

(i) Hyperbolic Parallel Property?

(ii) Euclidean Parallel Property?

(iii) Elliptic Parallel Property?

Which satisfy none of these properties?

2 Incidence and Betweenness

There are many models of geometry around us. If we start with any surface, we can take the points on the surface as our "points", and use the shortest paths between these points to define "lines". Point B is "between" A and C if the shortest path from A to C passes through B.

We will be interested in discovering the properties of these models. Our first inclination is to do this by studying models one by one, but this approach is extremely inefficient. Instead, we will adopt what is called the *axiomatic method*. The idea is simple. We will first identify a small list of properties that are common to the models we are interested in studying. These properties will be called *axioms*. We will then focus on discovering new properties that follow logically from the axioms. These will be called *propositions* and *theorems*.

The economy of this method lies in the fact that once we check that our axioms hold in a particular model, we are assured that every proposition or theorem that follows from those axioms also holds in that model. The beauty of mathematics lies in the realization that a few simple axioms can spawn a multitude of deep and complex propositions and theorems.

We begin with three *incidence axioms*:

I-1: If P and Q are two distinct points, then there is a unique line (denoted by $\overline{PQ}$) containing both P and Q.

I-2: Every line contains at least two distinct points.

I-3: There exist three distinct non-collinear points, X, Y, and Z.

I-1 is often abbreviated by saying that "Two points determine a line." Notice that $\overline{PQ} = \overline{QP}$. Axioms **I-2** and **I-3** serve only to eliminate trivial models. You can now prove that the following propositions hold in every model of the incidence axioms. Apart from the statements of **I-1**, **I-2**, and **I-3**, there is absolutely nothing else you need to know to construct these proofs.

Proposition 1: No point lies on every line.

Proposition 2: There exist 3 distinct lines with no point on all 3.

Proposition 3: Every point is on more than one line.

In addition to the three incidence axioms, we will use four axioms about betweenness. The first two of these axioms tell us what three collinear points look like.

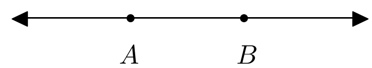
B-1 (Symmetry Axiom): If $[A, B, C]$, then A, B, and C are three distinct collinear points and $[C, B, A]$.

B-2 (Uniqueness Axiom): If A, B, and C are three distinct collinear points, then exactly one of them is between the other two.

From these two axioms alone, we can establish two facts about segments and rays.

Proposition 4: If A and B are distinct points, then

1. $\overleftarrow{AB} \cup \overrightarrow{AB} = \overline{AB}$
2. $\overleftarrow{AB} \cap \overrightarrow{AB} = AB$

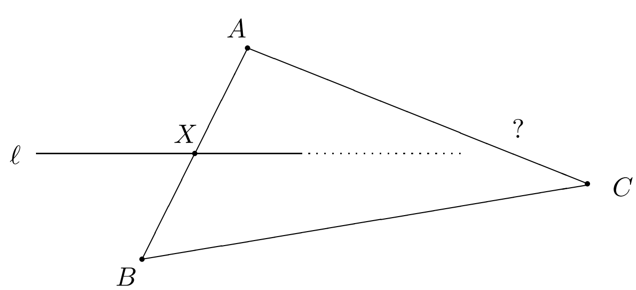


It will be useful for us to extend our notation for betweenness. For a finite set $A_1, A_2, \cdots, A_n$ of points, we will write $[A_1, A_2, \cdots, A_n]$ as an abbreviation for the conjunction of all statements $[A_i, A_j, A_k]$ where $i < j < k$. Thus $[A_1, A_2, \cdots, A_n]$ asserts that these are n distinct collinear points which lie in the order that they are written. The next axiom implies that every line has infinitely many points.

B-3 (Infinity Axiom): If B and D are distinct points, then there are distinct points A, C, and E on line $\overline{BD}$ such that $[A, B, C, D, E]$.

A *triangle* is the union of three segments AB, AC, and BC, where A, B, and C are non-collinear points. We denote this triangle by $\triangle ABC$. The points A, B, and C are called its *vertices*, and the three segments are called its *sides*.

Pasch's Theorem: Assume that line ℓ intersects side AB of $\triangle ABC$ at a point X between A and B. Then ℓ also intersects one of the other two sides. If C does not lie on ℓ, then ℓ does not intersect both AC and BC.



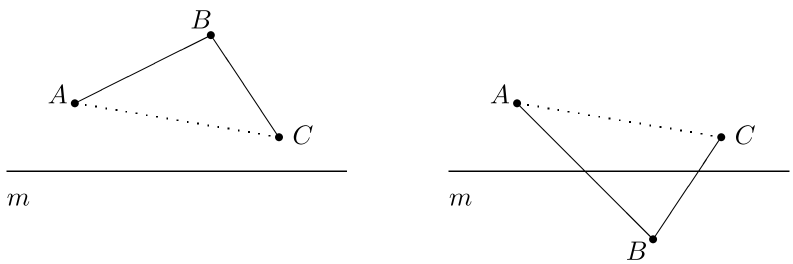
If you have difficulty proving this theorem, look back at the six models of Chapter 1. Is there one of them in which our axioms all hold, but Pasch's Theorem fails? If so, then we would know immediately that we couldn't possibly prove Pasch's Theorem from the axioms!

Our last betweenness axiom essentially says that the model is 2-dimensional. If m is a line and A and B are points not on m, we say that A and B are *on the same side of* m, written $A, B \mid_m$, if there is no point C on m such that $[A, C, B]$. When there is a point C on m such that $[A, C, B]$, we write $A \mid_m B$, and say that A and B are *on opposite sides of* m.

B-4 (Plane Separation Axiom):

(1) If $A, B \mid_m$ and $B, C \mid_m$, then $A, C \mid_m$.

(2) If $A \mid_m B$ and $B \mid_m C$, then $A, C \mid_m$.



This axiom says that every line separates the set of points into two "half planes", in the following sense:

Proposition 5: If m is a line, then there are two points A and B not on m, such that for every point C not on m, either $A, C \mid_m$ or $B, C \mid_m$, but not both.

Problem: Determine which of the models $\mathbb{R}^2$, $\mathbb{D}$, $\mathbb{P}$, $\mathbb{Q}^2$, $\mathbb{R}^3$, and $\mathbb{S}^2$ satisfy each of our seven axioms.

Here is a simple way to determine that two points are on the same side of a line.

Proposition 6: Suppose that $A \in \ell$, that $B, C \notin \ell$, and that $[A, B, C]$. Then $B, C \mid_\ell$.

Now you should be able to prove Pasch's Theorem. You can also prove the following two propositions; though they only concern points on a single line, they require the Plane Separation Axiom for their proofs. These two propositions tell us what four points on a line can look like.

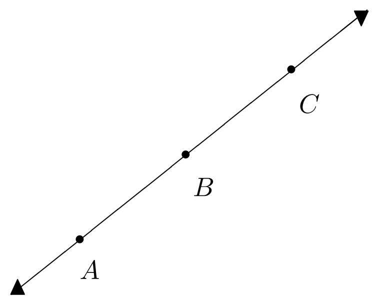
1st 4-Point Proposition: If $[A, B, C]$ and $[A, C, D]$, we have $[A, B, C, D]$.

2nd 4-Point Proposition: If $[A, B, C]$ and $[B, C, D]$, we have $[A, B, C, D]$.

The next two propositions are similar to Proposition 4, but require our additional axioms.

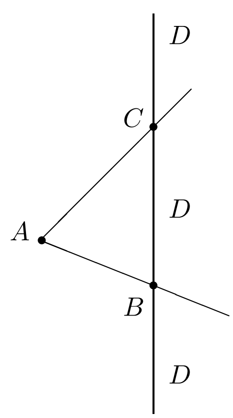
Proposition 7: If $[A, B, C]$, then

(1) $AB \cup BC = AC$
(2) $AB \cap BC = \{B\}$
(3) $\overleftarrow{AB} \cup \overrightarrow{BC} = \overline{AC}$
(4) $\overleftarrow{AB} \cap \overrightarrow{BC} = \{B\}$
(5) $\overrightarrow{AB} = \overrightarrow{AC}$



Rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are *opposite* rays if A and C are the same point and A is between B and D. Otherwise, they are *non-opposite rays*. Let A, B, and C be non-collinear points. The *angle* $\angle CAB$ is the union of the non-opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The *interior* of $\angle CAB$ is the set of points D that are on the same side of $\overline{AB}$ as C and the same side of $\overline{AC}$ as B.

Lemma 8: Let D be a point lying on line $\overline{BC}$. Then D is in the interior of $\angle CAB$ if and only if it is between B and C.



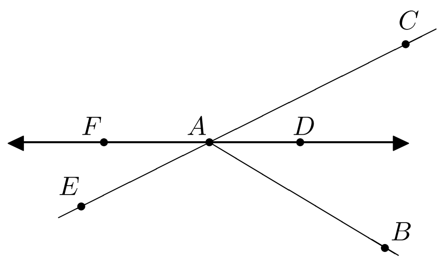
Lemma 9: Assume D is in the interior of $\angle CAB$, with $[EAC]$ and $[FAD]$. Then

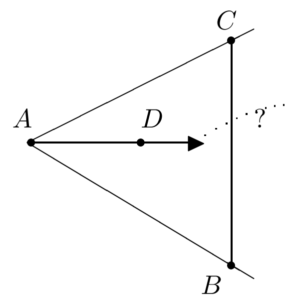
(1) every point on ray $\overrightarrow{AD}$ other than A is in the interior of $\angle CAB$;

(2) no point on the opposite ray $\overrightarrow{AF}$ is in the interior of $\angle CAB$;

(3) B and C are on opposite sides of $\overline{AD}$.

Ray $\overrightarrow{AD}$ is said to be *between* rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ if A, B, and C are not collinear, and D is in the interior of $\angle CAB$. (Notice that this definition would make no sense without part (i) of Proposition 9!)





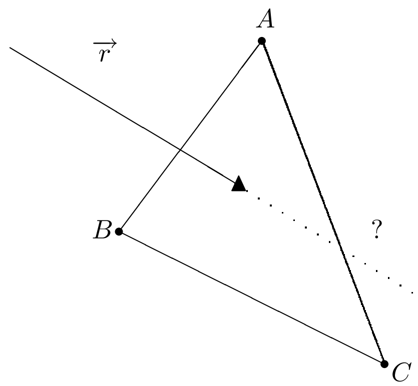
Crossbar Theorem: If $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\overrightarrow{AD}$ intersects the segment BC.

The *interior* of $\triangle ABC$ is the set of points that are simultaneously on the A side of BC, the B side of AC, and the C side of AB; that is, in the intersection of the three angle interiors. The *exterior* of the triangle consists of all points neither on the triangle nor in its interior.

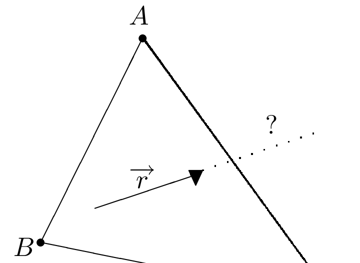
Proposition 10:

(1) If a ray $\overrightarrow{r}$ emanating from an exterior point of $\triangle ABC$ intersects side AB in a point between A and B, then $\overrightarrow{r}$ also intersects either side AC or side BC.

(2) If a ray $\overrightarrow{r}$ emanates from an interior point of $\triangle ABC$, then it intersects one of the sides; and if it does not pass through a vertex, then it intersects only one side.



A
$\overrightarrow{r}$
B
?
C



A
?
$\overrightarrow{r}$
B
C

3 Congruence

In this chapter, we study the notions of "congruence of segments" and "congruence of angles" in arbitrary models of geometry. These notions, along with "point", "line", and "betweenness", must be specified in each model. We will write

$$AB \cong CD$$

to mean that the segments AB and CD are congruent. In each of our models $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{Q}^2$, $\mathbb{D}$, and (locally) $\mathbb{S}$, we define segments AB and CD to be *congruent* if they have the same length, where the *length* of the segment connecting $P = (a, b)$ and $Q = (c, d)$ in the first four cases is $\left((a-c)^2 + (b-d)^2\right)^{1/2}$, and the *length* of PQ in $\mathbb{S}$ is measured along the shortest $\mathbb{S}$-line from P to Q. We will be interested in studying models of geometry in which the congruence axioms listed below are satisfied.

C-1 (Segment Duplication): Given segment AB and ray $\overrightarrow{CD}$, there is a unique point E on $\overrightarrow{CD}$ such that $AB \cong CE$.

C-2 (Addition of Segments): Suppose that $[A, B, C]$ and $[D, E, F]$. If $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

C-3 (Equivalence of Segments): For any segments AB, CD, and EF,

- $AB \cong AB$;
- if $AB \cong CD$, then $CD \cong AB$;
- if $AB \cong CD$ and $CD \cong EF$, then $AB \cong EF$.

Problem: Which of the segment congruence axioms hold in each of the models $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{Q}^2$, $\mathbb{D}$, and $\mathbb{S}$?

Proposition 11 (Subtraction of Segments): Suppose that $[A, B, C]$ and $[D, E, F]$. If $AB \cong DE$ and $AC \cong DF$, then $BC \cong EF$.

Proposition 12 (Division of Segments): Suppose that $[A, B, C]$ and $AC \cong DF$. Then there is a unique point E between D and F such that $AB \cong DE$ and $BC \cong EF$.

The congruence relation on segments gives us a natural way to order segments. We say that $AB < CD$ (or $CD > AB$) if there is a point E between C and D such that $AB \cong CE$.

Proposition 13 (Ordering of Segments): For points A, B, C, D, E, F with $A \neq B$, $C \neq D$, and $E \neq F$,

(1) $AB \not< AB$;

(2) if $AB > CD$ and $CD \cong EF$, then $AB > EF$;

(3) if $AB < CD$ and $CD \cong EF$, then $AB < EF$;

(4) if $AB < CD$ and $CD < EF$, then $AB < EF$;

(5) exactly one of the following is true: $AB < CD$, $AB \cong CD$, $AB > CD$.

Our last undefined notion is that of "congruence of angles". We denote the angle which is the union of the non-opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ by $\angle CAB$, or if the context makes it unambiguous, by $\angle A$. We write

$$\angle CAB \cong \angle C'A'B'$$

to indicate that the two angles are congruent. Combining segment and angle congruence leads to the defined notion of "triangle congruence": Triangles $\triangle ABC$ and $\triangle A'B'C'$ are *congruent*

$$\triangle ABC \cong \triangle A'B'C'$$

if under the correspondence $A \leftrightarrow A'$, $B \leftrightarrow B'$, and $C \leftrightarrow C'$ between their vertices, the angles $\angle A$, $\angle B$, and $\angle C$ are congruent, respectively, to the angles $\angle A'$, $\angle B'$, and $\angle C'$; and the sides AB, BC, and AC are congruent, respectively, to the sides $A'B'$, $B'C'$, and $A'C'$. Notice that the truth of the statement "$\triangle ABC \cong \triangle A'B'C'$" depends on the order in which the vertexes are listed!

In each of our models, we say that two angles are congruent if they have the same degree measure, where in $\mathbb{S}$, degrees are measured between the tangent lines. To our list of models, we now add a collection of new ones:

$\mathbb{B}$ **(the Bumpy Planes):** Begin with the plane $\mathbb{R}^2$, and imagine heating it like a pizza in the oven, so that the surface forms an irregular pattern of bumps and valleys. Points, lines, and betweenness are the same as in $\mathbb{R}^2$, but lengths and angle measures are changed. We say that two segments are congruent if they have the same length, and that two angles are congruent if they have the same degree measure.

Every bumpy plane satisfies the incidence and betweenness axioms, since points, lines, incidence, and betweenness are exactly the same as in $\mathbb{R}^2$. It is easy to check that the segment congruence axioms are also satisfied in $\mathbb{B}$.

Bumpy planes can be thought of as better models of irregular surfaces, like that of the earth, than $\mathbb{R}^2$. According to general relativity, space is not "flat" like $\mathbb{R}^2$, but rather is warped into "bumps" around massive objects. Though bumpy planes model physical reality better than does $\mathbb{R}^2$, this advantage comes with the price that they are generally much more complex than $\mathbb{R}^2$.

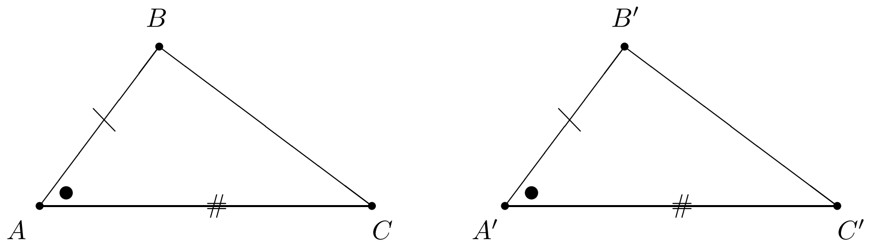
We now list our final axioms for how we like angle congruence to behave.

C-4 (Angle Duplication): Given angle $\angle CAB$ and ray $\overrightarrow{A'B'}$, there is a unique ray $\overrightarrow{A'C'}$ on a given side of line $\overline{A'B'}$ such that $\angle CAB \cong \angle C'A'B'$.

C-5 (Equivalence of Angles): For any angles $\angle A$, $\angle B$, and $\angle C$,

- $\angle A \cong \angle A$;
- if $\angle A \cong \angle B$, then $\angle B \cong \angle A$;
- if $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

C-6 (SAS): If two sides and the included angle of one triangle are congruent, respectively, to the corresponding two sides and included angle of another triangle, then the two triangles are congruent.



Notice that the conclusion of **C-6** is, by definition, an abbreviation for six assertions, so **C-6** tells us that three of those six imply the other three. In this sense, **C-6** is similar to the two Four Point Propositions.

Problem: Which of the angle congruence axioms are satisfied in each of our models? Which models satisfy all of the incidence, betweenness, and congruence axioms?

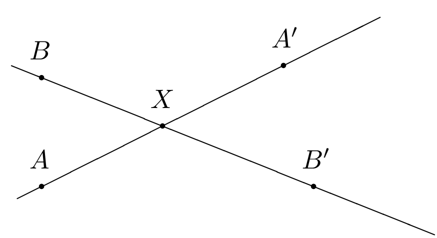
A triangle is said to be *isosceles* if two of its sides are congruent. The following proposition has a *very* short proof which is a direct application of **C-6**.

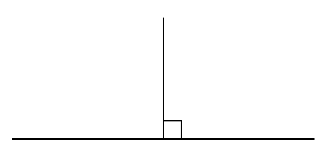
Proposition 14: The angles opposite to the congruent sides of an isosceles triangle are congruent.

A *supplement* of an angle $\angle CAB$ is an angle $\angle CAD$ where $[B, A, D]$.

Proposition 15: Congruent angles have congruent supplements.

If $[AXA']$ and $[BXB']$ but these points are not all collinear, then we say that the angles $\angle AXB$ and $\angle A'XB'$ are *vertical angles*. An angle that is congruent to one of its supplements is called a *right angle*.





Proposition 16: Vertical angles are congruent to each other.

Proposition 17: An angle congruent to a right angle is a right angle.

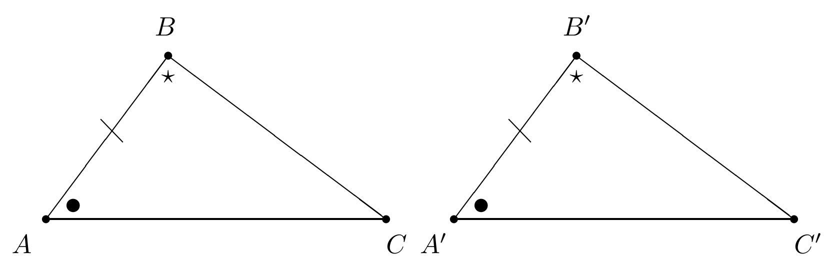
Proposition 18: Suppose that lines ℓ and m intersect at a point P. If one of the four angles formed is a right angle, then so are the other three.

Under the conditions of Proposition 18, we say that lines ℓ and m are *perpendicular.*

Proposition 19: Given a line ℓ and a point A, there is a line through A that is perpendicular to ℓ.

Question: Is the line described in Proposition 19 unique?

Theorem 20 (ASA Criterion for Congruence): If two angles and the included side of one triangle are congruent respectively to the corresponding two angles and included side of another triangle, then the two triangles are congruent.

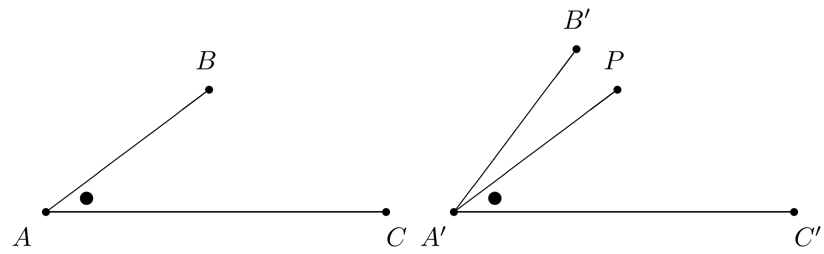


Proposition 21: If two angles of a triangle are congruent, then so are the two opposite sides (that is, the triangle is isosceles).

Proposition 22 (Angle Addition): Given point D in the interior of $\angle CAB$, and point D' in the interior of $\angle C'A'B'$, if $\angle CAD \cong \angle C'A'D'$ and $\angle DAB \cong \angle D'A'B'$, then $\angle CAB \cong \angle C'A'B'$.

Proposition 23 (Angle Subtraction): Given point D in the interior of $\angle CAB$, and point D' in the interior of $\angle C'A'B'$, if $\angle CAB \cong \angle C'A'B'$ and $\angle DAB \cong \angle D'A'B'$, then $\angle CAD \cong \angle C'A'D'$.

Ordering of angles is defined just as we defined ordering of segments. We say that $\angle CAB < \angle C'A'B'$ if there is a point P in the interior of $\angle C'A'B'$ such that $\angle CAB \cong \angle PA'B'$.



Proposition 24 (Ordering of Angles): For angles $\angle A$, $\angle B$, $\angle C$,

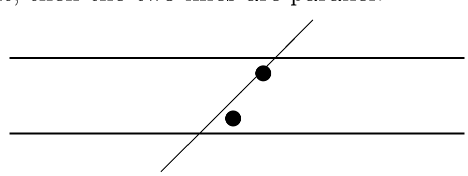
(1) exactly one of the following is true: $\angle A < \angle B$, $\angle A \cong \angle B$, $\angle A > \angle B$;
(2) if $\angle A < \angle B$ and $\angle B \cong \angle C$, then $\angle A < \angle C$;
(3) if $\angle A > \angle B$ and $\angle B \cong \angle C$, then $\angle A > \angle C$;
(4) if $\angle A < \angle B$ and $\angle B < \angle C$, then $\angle A < \angle C$.

This proposition gives us a useful way to prove two angles are congruent: assume that one is larger than the other, and show that this leads to a contradiction. We look at two applications of this idea.

Theorem 25 (SSS Criterion for Congruence): If the sides of one triangle are congruent respectively to the sides of another triangle, then the two triangles are congruent.

Proposition 26: Any two right angles are congruent to each other. (Hint: Use Proposition 24 and Proposition 9(iii).)

Theorem 27 (Alternate Interior Angle Theorem): If two lines intersect a third (transversal) line in such a way that the angles on opposite sides of the transversal and between the two lines (called *alternate interior angles*) are congruent, then the two lines are parallel.



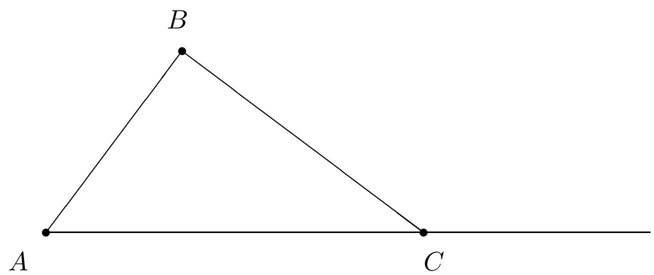
Notice that this theorem *does not* say the converse: if the lines are parallel, then the alternate interior angles must be parallel!

Proposition 28: Two lines perpendicular to the same line are parallel. In particular, if A is not on ℓ, then the perpendicular to ℓ through A is unique.

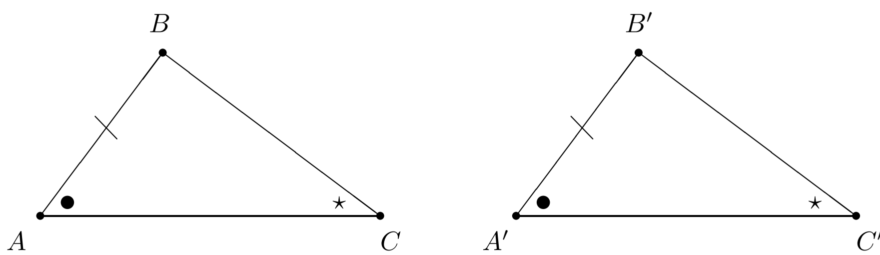
At the outset, we considered three possible ways in which parallel lines might behave uniformly, described by the Euclidean, Elliptic, and Hyperbolic Parallel Axioms. We can now show that our axioms prohibit one of these three possibilities.

Proposition 29 (Parallel Lines): Given a line ℓ, and a point P not on ℓ, there is at least one line through P parallel to ℓ.

Theorem 30 (Exterior Angle Theorem): An exterior angle (the supplement of an interior angle) of a triangle is greater than either remote interior angle.



Theorem 31 (SAA Criterion for Congruence): If two angles and the side opposite one in a triangle are congruent, respectively, to two angles and the side opposite one in another triangle, then the two triangles are congruent.



As anyone can plainly see, (ASS) is not a good criterion for congruence. However, there is a special case when it applies. A *right triangle* is a triangle containing a right angle. The *hypotenuse* of a right triangle is the side opposite the right angle, and the other two sides are called the *legs*.

Proposition 32 (HL Criterion for Congruence): Two right triangles are congruent if the hypotenuse and a leg of one are congruent to the hypotenuse and a leg of the other.

We say that P is a *midpoint* of segment AB if $[APB]$ where $AP \cong PB$. We say that the ray $\overrightarrow{AP}$ is a *bisector* of $\angle CAB$ if P is in the interior of $\angle CAB$ where $\angle CAP \cong \angle PAB$.

Proposition 33 (Midpoints): Every segment has a unique midpoint.

Proposition 34 (Bisectors): Every angle has a unique bisector.