Probability **FINAL EXAM** **Spring 2006**

Mathematical Systems **6/6/06**

- Please show your work on all problems. I like to give partial credit; try to make it easy for me to give it to you.
- You do not need to simplify any answers.
- This exam is closed-book and closed-notes. Naturally, no collaboration is allowed. However, you are allowed to use **TWO** handwritten 3x5 notecards!
- Each problem is worth 10 points.
- You have two hours to complete this exam.

The maximum possible score on this test is 100. Let's say that a score of 80 is perfect.

1. A drawer contains 20 socks, 10 fuchsia and 10 teal. You randomly select 4 socks. If 2 are fuchsia and 2 are teal then you stop; otherwise you replace them and randomly select 4 again. You continue doing this until you get 2 fuchsia and 2 teal socks. What is the expected number of selections you will have to make?

2. A test for inherent awesomeness is accurate 95% of the time on those who don't have it and 98% of the time on those who do (so it gives false positives 5% of the time and false negatives 2% of the time). If only .001% of the population is inherently awesome, what is the probability that a person is inherently awesome given that her test result is positive?

3. An airline offers two options for in-flight meals: Bag Of Cukes and Jedi Falafelwich. Experience shows that 70% of passengers prefer the cukes and 30% prefer the falafel. The airline takes on board 150 Bags Of Cukes and 80 Jedi Falafelwiches. Assume that there are 200 passengers and each makes his or her choice independently of the the others.

   (*Hint:* If you don't get the "Jedi Falafelwich" joke, pretend it's a lyric in an Alanis Morissette song. And you probably still won't get it.)

   (a) Find (but do not simplify or compute) the probability that each passenger gets his/her first choice of meal.

   (b) Find a good estimate of the chance that each passenger gets his/her first choice of meal.

4. The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. Suppose there are N floors above the ground floor, and each person is equally likely to get off at any one of these floors, independent of the number of people entering the elevator and where the others get off.

   (a) Compute the expected number of stops the elevator will make before discharging all of its passengers.

(b) If $N = 3$, find the joint distribution of X_1 and X_2, where X_i is the number of people who get off on the ith floor. Show that X_1 and X_2 are independent.

5. Let X and Y be continuous random variables having joint density function $f_{X,Y}(x, y) = \lambda^2 e^{-\lambda y}$ for $0 \leq x \leq y < \infty$ and $f_{X,Y}(x, y) = 0$ for all other pairs (x, y).

   (a) Find $f_X(x)$ and $f_Y(y)$. Recognize these as well-known distributions and say what the parameters are.

   (b) For a positive constant t, find $P\{Y - X > t\}$. What kind of distribution is $Y - X$?

6. At Super Saturday, Sean Connolly runs a stall where you get to throw a pie in his face. If you fail to hit him on the first try, you can try again, with a maximum of 3 throws. If one of your throws hits Sean's face, you get no more throws.

   Suppose n people play at Sean's stall. Assume that on each throw, person i hits Sean's face with probability p_i, independently of all other throws and all other people. Let T be the total number of pies that hit Sean in the face.

   (a) Find $E[T]$.

   (b) Find $Var(T)$.

7. Consider an urn containing a large number of coins, and suppose that each of the coins has some probability p of turning up heads when flipped, where p varies from coin to coin. Suppose that for a randomly selected coin, we can regard its p value as being the value of a random variable P that is uniformly distributed over $[0, 1]$.

   (a) If a coin is selected at random, find the probability that the first two tosses are heads.

   (b) Find the conditional distribution of P given that the first two tosses of the selected coin are heads.

   (c) If the first two tosses of the selected coin are heads, find the expected number of heads in the next four tosses.

8. The points 0, 1, 2, 3, and 4 lie in order on a line. A man is standing at either 1, 2, or 3. Every minute he takes a step to the point to his immediate left or his immediate right with probability .5 each; he continues until he reaches either point 0 or point 4. Let X_i denote how many minutes it takes him to reach either 0 or 4 if he starts at point i.

   (a) Show that the expected value of X_1 is 3. (*Hint:* Write equations for the expected value of X_i in terms of the expected values of X_{i-1} and/or X_{i+1} and solve the resulting system of equations.)

   (b) Repeat the above problem with 4 replaced by n and 3 replaced by $n - 1$. (*Sidebar:* It is interesting to consider what this implies when we let n go to infinity.)

9. You have a coin that, when flipped, comes up heads with probability p $(0 < p < 1)$ and tails with probability $1 - p$. Unfortunately, you don't know the value of p. Morever, what you *really* want is a fair coin.

Your power animal tells you you can simulate a fair coin as follows: Flip your coin twice. If the outcomes are HT, the simulated coin's outcome is H; if the outcomes are TH, the simulated coin's outcome is T. If the outcomes are HH or TT, repeat the process. Continue until you have an outcome for the simulated coin.

(a) Show that you are certain to get an outcome for the simulated coin.

(b) Show that the simulated coin has a probability of .5 of coming up H.

10. There are n envelopes, marked 1 through n, and n cards, also marked 1 through n. The cards are distributed randomly into the envelopes, one card per envelope. (So all $n!$ possible arrangements of cards in envelopes are equally likely.) Say a *match* occurs in envelope i if card i gets put into envelope i. Let M be the total number of matches that occur.

(a) Find $E[M]$.

(b) Find $Var(M)$.

"I think you're begging the question," said Haydock, "and I can see looming ahead one of those terrible exercises in probability where six men have white hats and six men have black hats and you have to work it out by mathematics how likely it is that the hats will get mixed up and in what proportion. If you start thinking about things like that, you would go 'round the bend. Let me assure you of that!

- Agatha Christie, *The Mirror Crack'd*