Non-Linear Dynamics Homework Solutions

Week 1

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March 5, 2007

Please email me at smachr09@evergreen.edu with any corrections or concerns reguarding these solutions.

2.3.2 We are given that $\dot{x} = k_1 a x - k_{-1} x^2$ models some chemical reaction.

a) Find all fixed points.

Fixed points occur where $\dot{x} = 0$. This happens when $0 = k_1 a x - k_{-1} x^2$. This implies that $x^* = 0$ or $x^* = (k_1 a)/(k_{-1})$.

By plugging in values of x inbetween and on each side of the two fixed points, we find that for $x < 0$, $\dot{x}$ is negative; for x such that $0 < x < (k_1 a)/(k_{-1})$ we have that $\dot{x} > 0$ and for $x > (k_1 a)/(k_{-1})$ we have that $\dot{x} < 0$. From this we can conclude that $x^* = 0$ is an unstable fixed point and $x^* = (k_1 a)/(k_{-1})$. is a stable fixed point.

b) Sketch the graph of $x(t)$ for various initial values x_0.

Figure 1 has been obtained using the specific values $k_1 a = 5$ and $k_{-1} = 2$, but any other values should give a similar diagram.

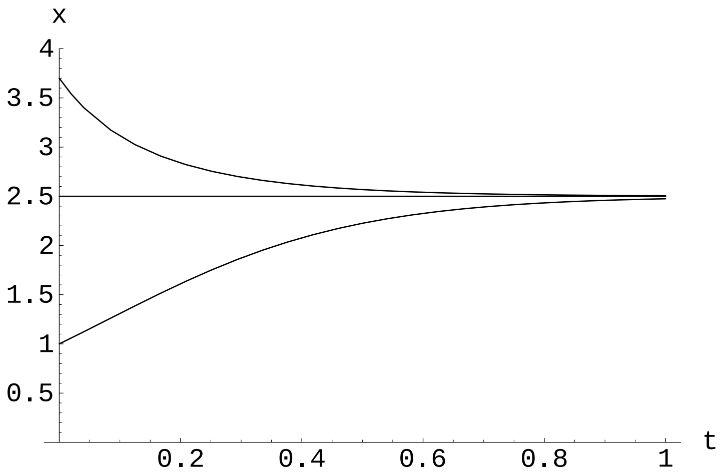

Figure 1: Solutions for Various Initial Values Given $k_1 a = 5$ and $k_{-1} = 2$

2.3.3 We study the Gompertz model of tumor growth given by $\dot{N} = -aN\ln(bN)$, where $N(t)$ is proportional to the number of cancer cells in the tumor and $a, b > 0$ are parameters.

a) Interpret a and b biologically.

Higher a values correspond to greater absolute values of $\dot{N}$ given any fixed values of b and N. Hence a can be thought of as a parameter which corresponds to the how quickly the population size changes given some fixed b value and an initial N value.

When $N > 1/b$, $\ln bN > 0$ so $\dot{N} < 0$. When $N < 1/b$, $\ln bN < 0$ so $\dot{N} > 0$. When $N = 1/b$, $\ln bN = 0$ so $\dot{N} = 0$.

From this we can conclude that $1/b$ is a stable fixed point, and will be the size of the tumor as $t \to \infty$ if you managed to live forever.

b) Sketch the vector field and graph $N(t)$ for various initial values.

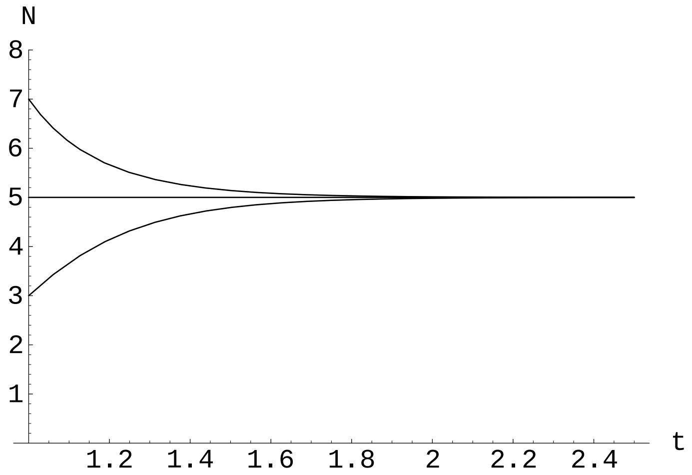

Figure 2: Graph of $N(t)$ for Various Initial Values and Specific a and b Values

For Section 2.4 you should use stability analysis and graphical analysis, as apropriate, to classify the stability of the fixed points of the systems in question.

2.4.4 $\dot{x} = x^2(6 - x)$

Clearly, the fixed points here are 0 and 6. To classify these fixed points, first we let $\dot{x} = f(x)$. Then $f'(x) = 3x(x - 4)$. Thus when $x = 6$, we have that $f'(x) > 0$, whence $x^* = 6$ is an unstable fixed point.

Our other fixed point is $x^* = 0$. Evaluated here, $f'(x) = 0$ and so we can conclude nothing about the stability of this fixed point using linear stability analysis. Using a graphical argument, we can see from Figure 3 that $x^* = 0$ is bistable.

2.4.7 $\dot{x} = f(x) = ax - x^3$ for some $a \in \mathbf{R}$. Discuss the cases where a is positive, negative and zero.

Well, $\dot{x} = x(a - x^2)$ so $x^* = 0$ or $x^* = \pm\sqrt{a}$.

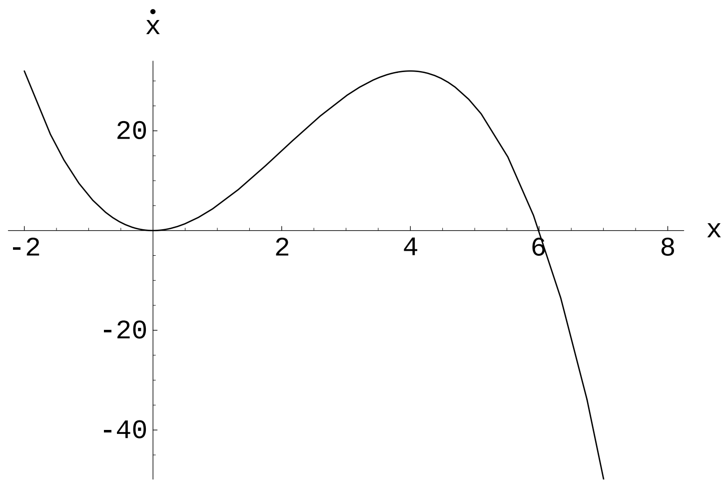

Figure 3: Plot of $\dot{x}$ vs. x

$\mathbf{a = 0}$ Then $\dot{x} = -x^3$ and so $x^* = 0$ is stable (this follows by checking the sign of points to either side of the equilibrium) and the only fixed point.

$\mathbf{a > 0}$ Then $f'(x) = a - 3x^2$, which is poitive at $x^* = 0$, indicating that this fixed point is unstable. Evaluating $f'(x)$ at $x^* = \pm\sqrt{a}$, we see that $f'(x) = -2a$ and is therefor negative. This implies that those fixed points are stable.

$\mathbf{a < 0}$ Then $f'(x)$ is negative at $x^* = 0$, so this fixed point is stable. The other fixed points are imaginary and therefor of no concern to us.

2.4.8 $\dot{N} = -aN\ln(bN)$, the Gompertz model from Problem 2.3.3.

If $\dot{N} = 0$ then $N = 0$ or $\ln bN = 0$ (since $a \neq 0$). However, $\ln 0$ is not defined, and in-class attempts to take the limit of $\dot{N}$ as $x \to \infty$ proved to be futile, so I'm not going to consider $N = 0$ further. Besides, behaviour is undefined for $N < 0$ so classifying the point isn't really possible.

Now if we assume that $\ln bN = 0$, we have that $N = 1/b$. We let$\dot{N} = f(N)$, and find that $f'(N) = -a(1 - \ln(bN))$. When evaluated at $N^* = 1/b$, we get $f'(N) = -a < 0$, from which we conclude that the only fixed point is a stable one, as we found before.

2.7.3 $\dot{x} = \sin x$

Since $\dot{x} = -dV/dx$, we can find V by mulitplying both sides by -1 and integrating with respect to x. Doing this we find that $V(x) = \cos x$. (Recall that the constant of integration can be set to zero for our purposes) We give the graph in Figure 4.

The set of fixed points is the set of values of x for which $V(x)$ takes on maxes or mins. This is easily seen to be the set $S = \{n\pi \in \mathbf{R} | n \in \mathbf{N}\}$. The stable points are those for which n is even and the unstable ones are those for which n is odd.

2.7.4 $\dot{x} = 2 + \sin x$

We find that $V(x) = \cos x - 2x$ using the same reasoning as we did in the last problem. Looking at Figure 5 we can see that there are no critical points and so there are no fixed points of the system.

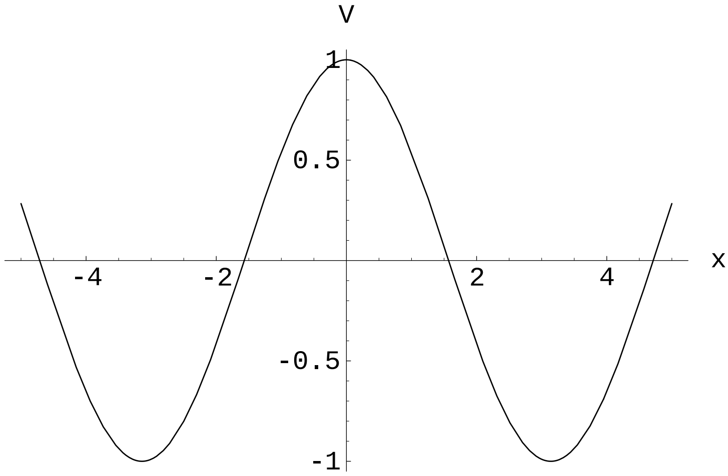

Figure 4: Plot of $V(x)$

3.1.1 $\dot{x} = 1 + rx + x^2$

To solve this problem, suppose that $\dot{x} = 0$ has real solutions a and b. Then we can write $\dot{x} = (x - a)(x - b)$. Since we we must have that $(x - a)(x - b) = 1 + rx + x^2$, we conclude that $ab = 1$, and that $a + 1/a = r$. From the first conclusion, we deduce that $b = 1/a$. Hence $\dot{x} = (x - a)(x - 1/a)$.

Analyzing the bifurcation, we first look at Figure 6, which is a 3-dimensional plot of $\dot{x}$ as a function of r and x and it's intersects with the zero plane. This intersection itself will be the shape of the bifurcation curve, and where we draw solid or dotted lines we can figure out from it as well by considering constant r slices and thinking about them as vector fields describing how x changes. To find the actual bifurcation point, note that at this point the vertex of the graph must lie on the x-axis. This is only possible for a polynomial if there is a repeated root, which would imply that $a = 1/a$ which is only possible if $a = 1$. Since $a + 1/a = r$, we must have that $r_c = 2$.

See Figure 7 for the bifurcation diagram.

3.1.3 $\dot{x} = r + x - \ln(1 + x)$

This system has fixed points when $r + x = \ln(1 + x)$. To find the set of points which satisfy this relation, we plot $y = r + x$ and $y = \ln(1 + x)$ on the same graph (for $r = 5$) and consider what happens as we change r (See Figure 8).

As r decreses, the line's y-intercept will also. We can now see that the curves intersect when $r \leq 0$. To see that $r = 0$ is in fact the bifurcation point, note that $y = r + x$ and $y = \ln(1 + x)$ have equal derivatives at the point $x = 0$, reguardless of the value of r. Thus since $y = x$ and $y = \ln(1 + x)$ intersect at the origin, have the same derivitve there and since $y = \ln(1 + x)$ has negative curvature everywhere, we get that when $r = 0$ there is exactly one intersection of the graphs. It is easily seen that for $r < 0$ there are two intersection between the graphs. Thus $r = 0$ marks the change from there being no fixed points to two and is therefore a blue sky bifurcation.

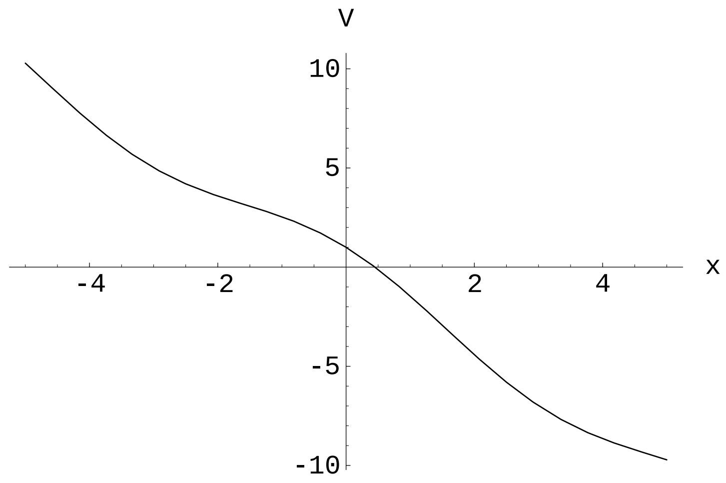

Figure 5: Plot of $V(x)$

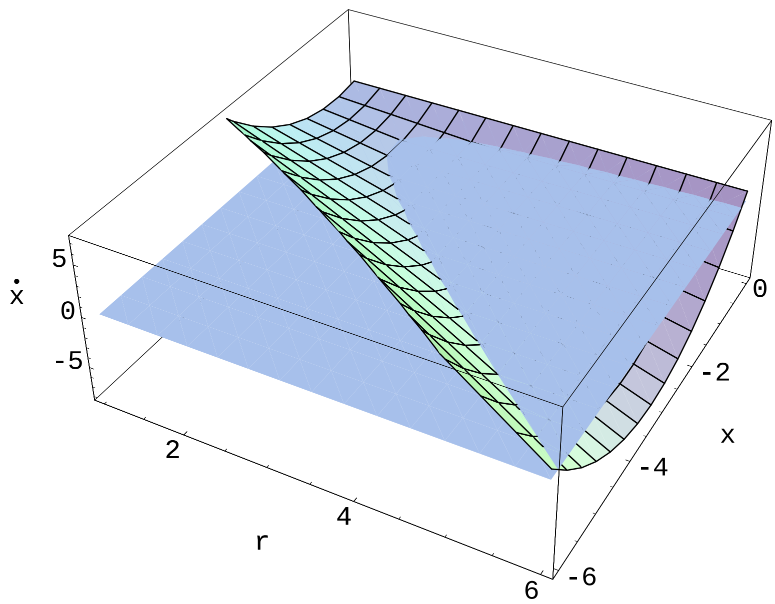

Figure 6: 3-D Plot of $\dot{x}$ vs r and x

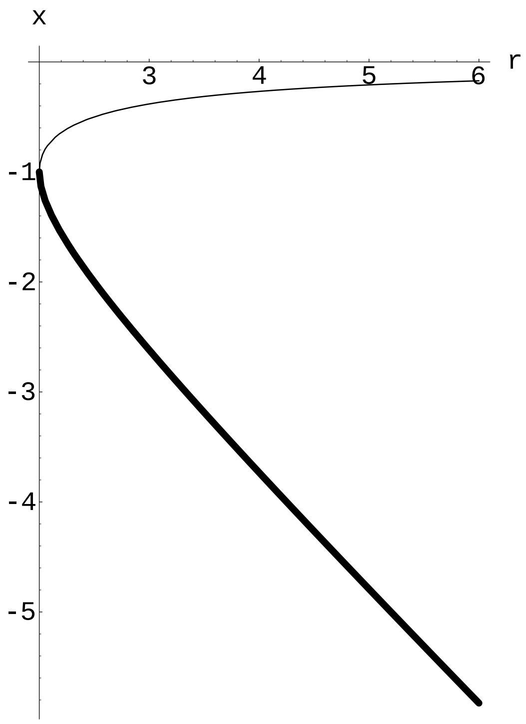

Figure 7: Bifurcation Diagram (here bold lines are stable, thin are unstable)

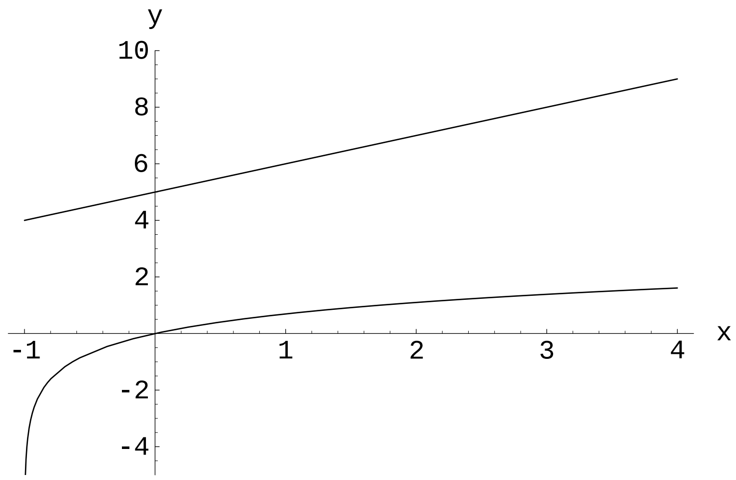

Figure 8: Plots of $y = r + x$ and $y = \ln(1 + x)$ for $r = 5$