

Solutions to PDE Handout (Week 1)

1. $u_{xx} + u = 0 \Rightarrow u_{xx} = -u$

Guess: $u = f(y)e^{\lambda x}$ from the Method of Undetermined Coefficients

so $u_x = f(y)\lambda e^{\lambda x}$ and $u_{xx} = f(y)\lambda^2 e^{\lambda x}$

plugging our guess for u and u_{xx} back into our equation gives

$$f(y)\lambda^2 e^{\lambda x} = -f(y)e^{\lambda x} \text{ which reduces to } \lambda^2 = -1 \text{ and } \lambda = i$$

$$\text{So } \boxed{u(x, y) = f(y)e^{ix} = f(y)(\cos(x) + i\sin(x)) = f(y)(a\cos(x) + b\sin(x))}$$

2. $u_x = \sin(xy) \Rightarrow u(x, y) = \int \sin(xy) dx = -\frac{1}{y} \cos(xy) + f(y)$

Given: $u(0, y) = y^2$

$$\text{So } u(0, y) = -\frac{1}{y} \cos(0) + f(y) = -\frac{1}{y} + f(y) = y^2$$

$$\Rightarrow f(y) = y^2 + \frac{1}{y}$$

and $u(x, y) = -\frac{1}{y} \cos(xy) + y^2 + \frac{1}{y}$

$$\text{Simplifying } \Rightarrow \boxed{u(x, y) = y^2 + \frac{1}{y}(1 - \cos(xy))}$$

3. a) $A=1$ $B=1$ $C=0$ $B^2 - 4AC = 1 > 0 \Rightarrow$ hyperbolic PDE

b) $A=1$ $B=0$ $C=-1$ $B^2 - 4AC = 4 > 0 \Rightarrow$ hyperbolic PDE

c) $A=1$ $B=0$ $C=3$ $B^2 - 4AC = -12 < 0 \Rightarrow$ elliptic PDE

d) $A=9$ $B=-6$ $C=1$ $B^2 - 4AC = 0 \Rightarrow$ parabolic PDE

e) $A=1$ $B=0$ $C=\frac{1}{r^2}$ $B^2 - 4AC = -\frac{4}{r^2} < 0 \Rightarrow$ elliptic PDE

$$4. u_{tt} = u_{xx} \Rightarrow u_{tt} - u_{xx} = 0 \quad A=1 \quad B=0 \quad C=-1 \quad B^2 - 4AC = 4 > 0$$

This is a hyperbolic PDE. We expect a general solution in the form

$$u(x,t) = f(x+\lambda_1 t) + g(x+\lambda_2 t)$$

$$\text{From } A+B\lambda+C\lambda^2=0 \text{ we get } 1-\lambda^2=0$$

$$\lambda^2=1 \Rightarrow \lambda_1=1, \lambda_2=-1$$

$$\text{General Solution: } u(x,t) = f(x+t) + g(x-t)$$

$$\text{Given initial conditions: } u(x,0) = \sin(x) \text{ and } u_t(x,0) = x$$

$$u(x,0) = f(x+0) + g(x+0) = f(x) + g(x) = \sin(x)$$

$$\text{and } f'(x) + g'(x) = \cos(x) \quad (1)$$

$$u_t(x,t) = f'(x+t) - g'(x-t) \text{ and } u_t(x,0) = f'(x) - g'(x) = x$$

$$\Rightarrow f'(x) = x + g'(x) \quad (2)$$

$$\text{Substituting (2) into (1)} \Rightarrow x + g'(x) + g'(x) = \cos(x)$$

$$\Rightarrow 2g'(x) = \cos(x) - x \Rightarrow g'(x) = \frac{1}{2}\cos(x) - \frac{1}{2}x$$

$$g(x) = \int \left(\frac{1}{2}\cos(x) - \frac{1}{2}x \right) dx = \frac{1}{2}\sin(x) - \frac{1}{4}x^2 + k$$

$$f(x) + g(x) = \sin(x) \Rightarrow f(x) + \frac{1}{2}\sin(x) - \frac{1}{4}x^2 + k = \sin(x)$$

$$\Rightarrow f(x) = \sin(x) - \frac{1}{2}\sin(x) + \frac{1}{4}x^2 - k = \frac{1}{2}\sin(x) + \frac{1}{4}x^2 - k$$

$$\text{Finally, } u(x,t) = \frac{1}{2}\sin(x+t) + \frac{1}{4}(x+t)^2 - k + \frac{1}{2}\sin(x-t) - \frac{1}{4}(x-t)^2 + k$$

$$u(x,t) = \frac{1}{2}(\sin(x+t) + \sin(x-t)) + \frac{1}{4}((x+t)^2 - (x-t)^2)$$

5. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$ $A=1$ $B=-3$ $C=2$ $B^2 - 4AC = 1 > 0$
 This is a hyperbolic PDE. We expect a general solution in the form
 $u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$

From $A + B\lambda + C\lambda^2$ we get $1 - 3\lambda + 2\lambda^2 = 0$
 $\lambda = \frac{3 \pm \sqrt{1}}{4} = \frac{3 \pm 1}{4} \Rightarrow \lambda_1 = 1, \lambda_2 = \frac{1}{2}$

General Solution: $u(x, y) = f(x + y) + g(x + \frac{1}{2}y)$

Given boundary conditions: $u(x, 0) = -x^2$ and $u_y(x, 0) = 0$

$$u(x, 0) = f(x) + g(x) = -x^2 \quad \text{and} \quad f'(x) + g'(x) = -2x \quad (1)$$

$$u_y(x, y) = f'(x + y) + \frac{1}{2}g'(x + \frac{1}{2}y) \quad \text{and} \quad u_y(x, 0) = f'(x) + \frac{1}{2}g'(x) = 0$$

$$\text{and} \quad f'(x) = -\frac{1}{2}g'(x) \quad (2)$$

$$\text{Substituting (2) into (1)} \Rightarrow -\frac{1}{2}g'(x) + g'(x) = -2x$$

$$\Rightarrow \frac{1}{2}g'(x) = -2x \Rightarrow g'(x) = -4x$$

$$g(x) = \int -4x \, dx = -2x^2 + k$$

$$f(x) + g(x) = -x^2 \Rightarrow f(x) - 2x^2 + k = -x^2$$

$$\Rightarrow f(x) = 2x^2 - x^2 - k = x^2 - k$$

$$u(x, y) = (x + y)^2 - k - 2(x + \frac{1}{2}y)^2 + k$$

$$u(x, y) = (x + y)^2 - 2(x + \frac{1}{2}y)^2$$