Variational Calculus Homework Solutions - Assignment 1

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Please email me at smachr09@evergreen.edu with any corrections or concerns reguarding these solutions.

1. Find and solve the Euler-Lagrange equations for the following functional

$$I[y] = \int_{x_1}^{x_2} (y')^2 - y dx$$

We use the Euler-Lagrange equation to obtain the equality

$$-1 - \frac{d}{dx} 2y' = 0.$$

Since $\frac{d}{dx} 2y' = 2y''$, we can substitute the right hand side into our equality. Simplifying this gives

$$y' = -\frac{1}{2}.$$

To solve the differential equation it's easiest to wager the guess

$$y(x) = -\frac{1}{4}x^2 + ax + b$$

for some $a, b \in \mathbf{R}$. It is staightforward to check that this equation satisfies the differential equation. What's more, it has two parameters and thus solutions of this form span the solution space (Recall that the dimension of the solution space is equal to the order of differential equation you are working with).

2. The following problems are from Perfect Form.

2.1 We find the function $y(x)$, having boundary conditions $y(0) = 0$ and $y(1) = 1$, that makes the following integral stationary.

$$\int_{x_1}^{x_2} (y'^2 + yy' + y^2) dx$$

We use the normal Euler Lagrange equation (not the first integral, which gets messy) to get to the equation

$$0 = y' + 2y - \frac{d}{dx}(2y' + y)$$

which gives us $y'' = y$. We wager the guess $y(x) = e^{\lambda x}$, which would imply the equation $\lambda^2 = 1$, which gives us the linearly independent solutions $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. The general solution thus has the form $y(x) = Ay_1(x) + By_2(x)$. Our boundary conditions give us the equations $B = -A$ and $1 = Ae + Be^{-1}$. Substituting the first into the latter we get $A = 1/(e - e^{-1})$. Now

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

so $e - e^{-1} = 2\sinh 1$ and thus $A = 1/(2\sinh 1)$ implying that $B = -1/(2\sinh 1)$. Plugging this back into the equation for $y(x)$ we get the equation

$$\begin{aligned} y(x) &= \frac{e^x - e^{-x}}{2\sinh 1} \\ &= \frac{1}{\sinh 1}\frac{e^x - e^{-x}}{2} \\ &= \frac{\sinh x}{\sinh 1} \end{aligned}$$

2.3 *Minimum Surface* We show that the functional

$$A = \int_{x_1}^{x_2} 2\pi x\sqrt{1 + y'^2}dx$$

is minimized by the function

$$y(x) = C_1 \cosh^{-1}(x/C_1) + C_2.$$

We take the first integral $\partial f/\partial y' = c$ where f is the function on the inside of the integral and c is some constant. Taking this integral we get the equation

$$2\pi x y'(1 + y'^2)^{-1/2} = c.$$

Solving for y', we get

$$y' = \frac{\frac{1}{2\pi x}}{\sqrt{1 - \frac{1}{(2\pi x)^2}}}.$$

Integrating gives us a nasty integral which I solved using the substitution $u = \frac{1}{2\pi x}$ and *Table of Integrals, Series and Products, 5th Edition* and this comes out to exactly what it was supposed to. Details are painfull are will be excluded.

2.5 *Variational vs. Direct Method*

a) We show using the Euler-Lagrange Equation that the integral

$$I = \int_0^1 \left((y')^2 - \frac{\pi^2 y^2}{4}\right) dx$$

is made stationary by $y(x) = \sin(\pi x/2)$ given the initial conditions $y(0) = 0$ and $y(1) = 1$.

Applying the Euler-Lagrange equations we get that

$$\frac{\pi^2}{2}y - 2y'' = 0.$$

We rearange this to get

$$y'' = \frac{\pi}{2}^2 y,$$

which easily leads to the general solution

$$y(x) = C\sin(\frac{\pi}{2}x + D).$$

From here we note that $D = 0$ and $C = 1$ satisfy the conditions nicely, giving us the desired solution.