Non-Linear Dynamics Homework Solutions

Week 3

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Please email me at smachr09@evergreen.edu with any questions or concerns reguarding these solutions.

4.1.2 Find and classify all of the fixed points of

$$\dot{\theta} = 1 + 2\cos\theta$$

and sketch the phase portrait on the circle.

Fixed points occur at values θ^* such that $0 = 1 + 2\cos\theta^*$. Since this system is periodic with period 2π, we only need to consider values of θ^* inbetween 0 and 2π. Throughout one full period, $2\cos\theta$ varies continuously from 2 to -2 and then back to 2 at the end of the period. Thus, once on the way to -2 and once coming from -2 back up, $\cos\theta = -1$ will be true and consequently, $\dot{\theta} = 0$. Since $\dot{\theta} = -1$ when $\theta = \pi$, We must have the first fixed point before that value and the second one after. We also know that when $\theta = \pi/2$ or $3\pi/2$, $\dot{\theta} = 1$, so we let θ_1^* be the fixed point lying in the second qudrant and θ_2^* be the fixed point lying in the third.

For $\theta > 0$ but less that θ_1^*, we note that $\dot{\theta}$ is positive. For θ between θ_1^* and θ_2^*, $\dot{\theta}$ is negative. This means that θ_1^* is a stable fixed point. From what we said above, along with the observation that for $\theta > \theta_2^*$ but less than 2π, $\dot{\theta} > 0$, we have that θ_2^* is an unstable fixed point. Peicing this information together we get the following phase diagram on the circle.

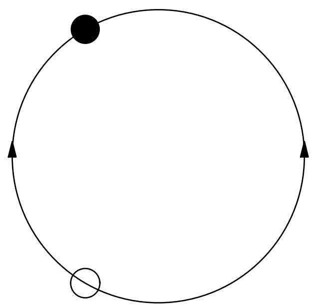

Figure 1: Vector Field on a Circle

4.2.3 High-School Chestnut

Let θ_h and θ_m be the angles that the hour and minute hands respectively make with 12:00. The hour hand moves at a constant rate of 2π radians every twelve hours, so $\dot{\theta_h} = \pi/6$. Similarly, we find that $\dot{\theta_m} = 2\pi$. As in exercise 4.2.1, we let $\phi = \theta_m - \theta_h$ be the phase difference between θ_h and θ_m. Then $\dot{\phi} = \dot{\theta_m} - \dot{\theta_h} = \frac{11}{6}\pi$. Since period T is given by $T = 2\pi/\omega$, where ω is the corresponding angular velocity, we have that the period is 12/11 hr.

As an alternative method, we could instead let $\theta_h(t) = \pi/6 \cdot t$ and $\theta_h(t) = 2\pi \cdot t$. Defining the phase difference ϕ as before, we get that $\phi(t) = \theta_m(t) - \theta_h(t) = \frac{11}{6}\pi \cdot t$, from which we obtain, as before, that $T = 12/11$ of an hour. (Do you feel jipped on the alternative appoach? I know I do. They are almost exactly the same. One simply talks about the derivative of a given function and the other more directly about the actual function.)

I personally like Pete's solution. He stared at a clock for a long time.

4.3.3 We consider the system

$$\dot{\theta} = \mu \sin\theta - \sin 2\theta.$$

To find the fixed points of the system we set $\dot{\theta} = 0$ and find that $\mu \sin\theta = 2\sin\theta\cos\theta$ by the double angle formula. Then either $\sin\theta = 0$ implying that $\theta^* = 0$ or π, or we can divide by $\sin\theta$, in which case we must have that $\mu/2 = \cos\theta^*$. It should now be apparent that for different values of μ will have different numbers of fixed points. Figure 2 gives a 3-D plot of $\dot{\theta}$ as a function of μ and θ.

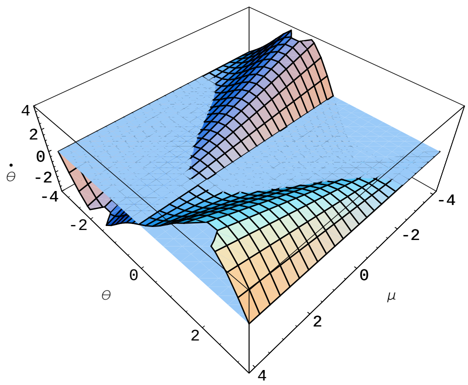

Figure 2: 3-D Plot of $\dot{theta}$ vs. θ and μ

From this we can extract a bifurcation diagram by considering it's intersection with the $\dot{\theta} = 0$ plane. The curves of intersection will form the shape of the bifurcation diagram and we can get stability information by considering the sign of $\dot{\theta}$ on each side of the bifurcation curve. Doing all this we get the bifurcation diagram of Figure 3 and can see that we have pitchfork bifurcations at the critical values of $\mu_c = \pm 2$. Note that we can also obtain these μ_c values by noting that solitary solutions to the equation $\mu/2 = \cos\theta^*$ first appear at these points and vanish, that two solutions are present for intermediate values, and that no solution exist for μ values outside of the interval.

Using all of this information we get the phase portraits of Figure 4 as we vary μ.

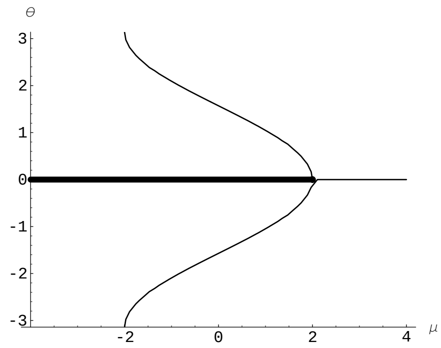

Figure 3: Bifurcation Diagram

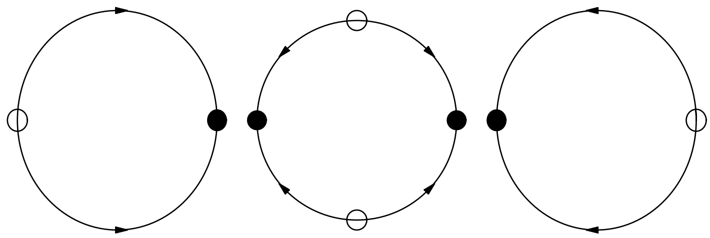

Figure 4: Vector Field as μ is Varied From Below -2 to Above 2

4.3.7 We consider the system given by

$$\dot{\theta} = \frac{\sin\theta}{\mu + \sin\theta}.$$

For this system we get fixed points exactly when $\sin\theta = 0$, that is when $\theta = 0$ or π, unless $\mu = 0$. In this case, $\dot{\theta} = 1$ for all θ (note that technically we might have to think about some limits here, but things still work out). For $\mu \neq 0$ such that $-1 \leq \mu \leq 1$ the denomenator of our expression for $\dot{\theta}$ will have zero values that do not correspond to zero values of the numerator, and thus we see that our system blows up in finite time for certian values of μ and θ. One of these blow up points emerges as μ approaches ± 1 and that as μ moves in from there the blow up point splits, leaving us with two blow up points. These blow up points aren't fixed points, so there's no real bifurcation going on here, but there is certainly weird behaviour emerging as we vary μ. To get a full picture of this behavior, observe both Figures 5 and 6.

4.4.4 *Torsion Spring* We study the equation of motion

$$b\dot{\theta} + mgL\sin\theta = \Gamma - k\theta$$

a) Does this system correspond to a well defined vector field?

Not unless $k = 0$. This becomes partiucularly apparent when noting that

$$\dot{\theta} = \frac{\Gamma - mgL\sin\theta}{b} - \frac{-k}{b}\theta.$$

The $k\theta$ term is not periodic but the term on the left of the sum is with period 2π. From this fact the system is periodic iff $k = .0$

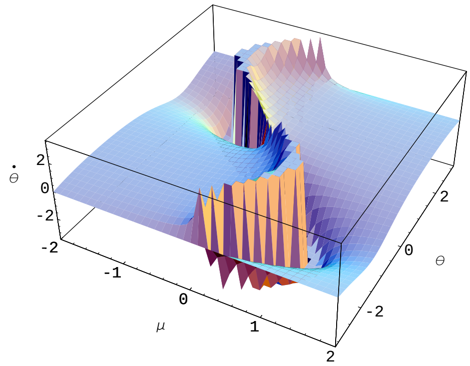

Figure 5: 3D Plot

b) We nondimensionalize the system.

We want to define our dimensionless units so that things simplify. With this in mind, we define $t = b/(mgL)\tau$. Then $dt/d\tau = b/mgL$ so that by the chain rule

$$\frac{d\theta}{d\tau} = \dot\theta\frac{dt}{d\tau} = \frac{\Gamma}{mgL} - \sin\theta - \frac{k}{mgL}\theta.$$

We shall further simplify this by letting $c = k/(mgL)$ and $\Gamma' = \Gamma/(gmL)$ So that we now have the equation

$$\frac{d\theta}{d\tau} = \Gamma' - c\theta - \sin\theta.$$

This checks out as being dimensionless. Since $\sin\theta$ is a summand, and is dimensionless, the other summands must be as well.

c) The system must be stable overall. You won't end up catching a ride to infinity from any starting point in this sytem, since we assume that $k \geq 0$. Under the assumption of $k > 0$, the value of the line $\Gamma' - c\theta$, will be much greater than $-\sin\theta$ for all θ less than some sufficiently negative value. The idea is supported by the illustrations of Figure 7.

d) Figure 7 displays these intersections for the same Γ' values but different c values. It becomes clear from considering the diagrams that as the slope of the line aproaches zero from below, infintiely many intersections arise, and the nature of these intersections implies that we have *blue-sky bifurcations* arising at these points.

4.5.3 *Exciteable Systems* We study a simple model of an exciteable system given by

$$\dot\theta = \mu + \sin\theta,$$

where μ is very close to 1.

a) To show that the system is exciteable, we identify the gloablally attracting rest state and system threshhold. First we note that this system is periodic and so defines a vector field on the unit circle. We show this in Figure 8.

From this we can see that the globally attracting resting state lies just to the left of $3\pi/2$ on the unit circle. An unstable fixed point lies just on the other side of $3\pi/2$ from the

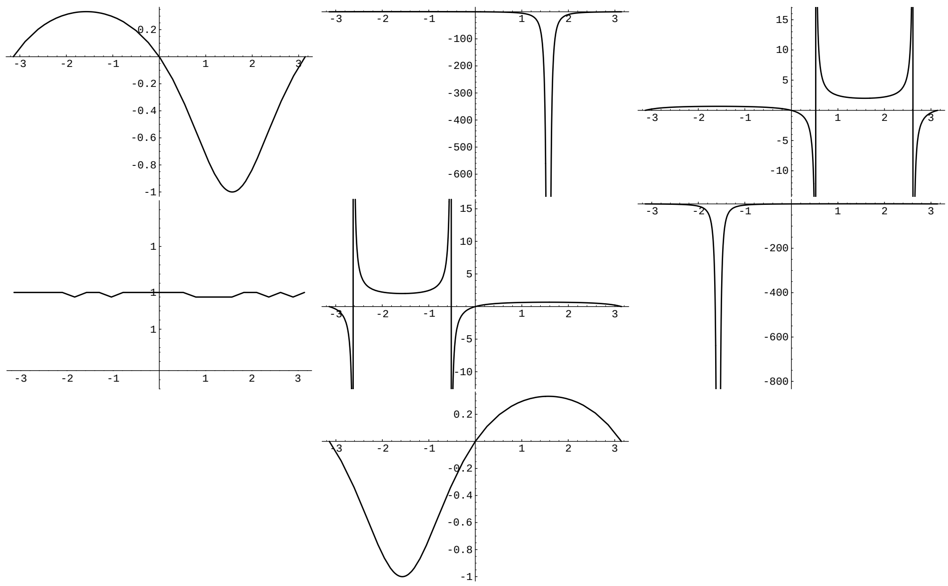

Figure 6: Phase Portraits as μ is Varied From -2 to 2

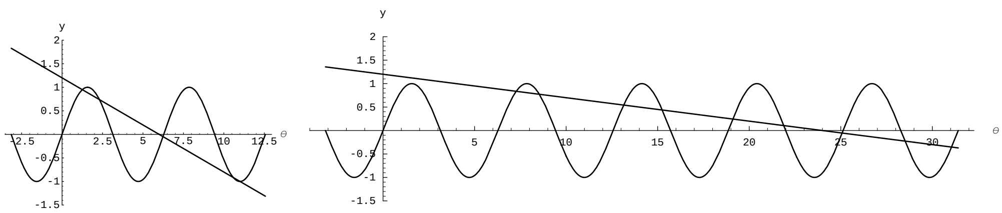

Figure 7: Intersections

stable fixed point. If for some reason θ is pushed just past this unstable fixed point, the flow of the sytem will be positive and so instead of decreasing back down to the rest state it will increase all the way around the circle to get back to the rest state.

b) I'm having difficulty in getting the graphics up for this one. Basically though, $\cos(\theta(t))$ gives the x coordinate of θ as it moves along the unit circle, so if we think about a reasonable path around the unit circle using the vector field, then if we just think about the motion of the x-coordinate, we get the behavior of the $V(t)$, the membrane potential.

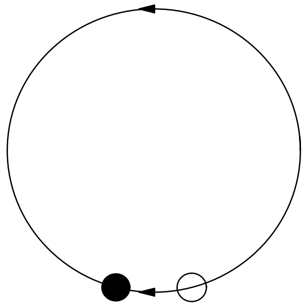

Figure 8: Vector Field