

PDE Solutions from Week 3

2.3.1.a $\frac{\partial u}{\partial t} = k \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$

let $u(r,t) = g(t)\phi(r)$ substituting for $u \Rightarrow \frac{\partial (g(t)\phi(r))}{\partial t} = k \frac{\partial}{\partial r} \left(r \frac{\partial (g(t)\phi(r))}{\partial r} \right)$

$\Rightarrow \phi(r) \frac{\partial g(t)}{\partial t} = k \frac{\partial}{\partial r} \left(r g(t) \frac{\partial \phi(r)}{\partial r} \right)$ divide both sides by $kg(t)\phi(r)$

$\Rightarrow \frac{1}{kg(t)} \frac{\partial g(t)}{\partial t} = \frac{1}{r \phi(r)} \frac{\partial}{\partial r} \left(r \frac{\partial \phi(r)}{\partial r} \right) \Rightarrow \frac{1}{kg} \frac{dg}{dt} = \frac{1}{r \phi} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) = -\lambda$

$\Rightarrow \frac{dg}{dt} = -kg\lambda$ and $\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) = -\lambda \phi$

2.3.1.b $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x}$

let $u(x,t) = g(t)\phi(x) \Rightarrow \frac{\partial (g(t)\phi(x))}{\partial t} = k \frac{\partial^2 (g(t)\phi(x))}{\partial x^2} - v_0 \frac{\partial (g(t)\phi(x))}{\partial x}$

$\Rightarrow \phi \frac{\partial g}{\partial t} = kg \frac{\partial^2 \phi}{\partial x^2} - v_0 g \frac{\partial \phi}{\partial x}$ divide both sides by $g\phi$

$\Rightarrow \frac{1}{g} \frac{dg}{dt} = k \frac{d^2 \phi}{\phi dx^2} - \frac{v_0}{\phi} \frac{d\phi}{dx} = -\lambda$

$\Rightarrow \frac{dg}{dt} = -\lambda g$ and $k \frac{d^2 \phi}{dx^2} - v_0 \frac{d\phi}{dx} + \lambda \phi = 0$

2.3.1.f $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

let $u(x,t) = g(t)\phi(x) \Rightarrow \frac{\partial^2 (g(t)\phi(x))}{\partial t^2} = c^2 \frac{\partial^2 (g(t)\phi(x))}{\partial x^2}$

$\Rightarrow \phi \frac{\partial^2 g}{\partial t^2} = gc^2 \frac{\partial^2 \phi}{\partial x^2}$ divide by $g\phi c^2$

$\Rightarrow \frac{1}{gc^2} \frac{d^2 g}{dt^2} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda$

2.3.2.d $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ B.C. $\phi(0) = 0$ $\frac{d\phi}{dx}(L) = 0$

guess $\phi = e^{rx} \Rightarrow \phi' = re^{rx}$ and $\phi'' = r^2e^{rx}$

substituting $\Rightarrow r^2e^{rx} + \lambda e^{rx} = 0 \Rightarrow r^2 = -\lambda$

for $\lambda > 0$

$r = \pm i\sqrt{\lambda} \Rightarrow \phi = ae^{i\sqrt{\lambda}x} + be^{-i\sqrt{\lambda}x} \Rightarrow \phi = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$

$\phi(0) = a\cos(0) + b\sin(0) = a + 0 = 0 \Rightarrow a = 0$

$\Rightarrow \phi = b\sin(\sqrt{\lambda}x)$ and $\phi' = \sqrt{\lambda}b\cos(\sqrt{\lambda}x)$

$\phi'(L) = \sqrt{\lambda}b\cos(\sqrt{\lambda}L) = 0$ $b \neq 0, \lambda \neq 0 \Rightarrow \cos(\sqrt{\lambda}L) = 0$

$\Rightarrow (\sqrt{\lambda}L) = (n - \frac{1}{2})\pi$

$\Rightarrow \sqrt{\lambda} = \left(\frac{(n - \frac{1}{2})\pi}{L}\right) \Rightarrow \lambda^2 = \left(\frac{(n - \frac{1}{2})\pi}{L}\right)^2 \quad n = 1, 2, 3, \dots$

for $\lambda = 0$

$\phi = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x) = a\cos(0) + b\sin(0) = a$

but $\phi(0) = a = 0 \Rightarrow a = 0 \Rightarrow$ trivial solution

for $\lambda < 0$

$r = \pm\sqrt{\lambda'} \Rightarrow \phi = ae^{\sqrt{\lambda'}x} + be^{-\sqrt{\lambda'}x}$

$\phi(0) = ae^0 + be^0 = a + b = 0 \Rightarrow b = -a$

so $\phi = ae^{\sqrt{\lambda'}x} - ae^{-\sqrt{\lambda'}x}$ and $\phi' = \sqrt{\lambda'}ae^{\sqrt{\lambda'}x} + \sqrt{\lambda'}ae^{-\sqrt{\lambda'}x}$

and $\phi'(L) = \sqrt{\lambda'}ae^{\sqrt{\lambda'}L} + \sqrt{\lambda'}ae^{-\sqrt{\lambda'}L} = 0$ iff $a = 0 \Rightarrow$ trivial solution

2.3.3.e $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ B.C. $u(0, t) = 0$ I.C. $u(x, 0) = 2\cos\left(\frac{3\pi x}{L}\right)$
 $u(L, t) = 0$

let $u = g(t)\phi(x) \Rightarrow \frac{d}{dt} \frac{dg}{dt} = k \frac{d}{dx} \frac{d\phi}{dx} \Rightarrow \frac{1}{k} \frac{dg}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda$

$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \Rightarrow \phi = e^{rx} \Rightarrow r^2 + \lambda = 0 \Rightarrow \phi = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$

$\frac{dg}{dt} = -\lambda k g \Rightarrow g = Ae^{-\lambda kt}$

so $u(x, t) = (a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x))Ae^{-\lambda kt}$

$u(0, t) = a\cos(0) = 0 \Rightarrow a = 0$

$u(L, t) = b\sin(\sqrt{\lambda}L)Ae^{-\lambda kt} = 0$

$\Rightarrow \sin(\sqrt{\lambda}L) = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \dots$

$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$

where $B_n = \frac{2}{L} \int_0^L 2\cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$ (equation 2.3.40)

2.3.4.a $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ B.C. $u(0,t) = 0$ I.C. $u(x,0) = f(x)$
 $u(L,t) = 0$

total heat energy $\Rightarrow \int_0^L e(x,t) dx = \int_0^L c \rho A u(x,t) dx$

from 2.3.3.c we have $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$

$$\int_0^L e(x,t) dx = c \rho A \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} dx = c \rho A \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= c \rho A \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \left[\frac{-L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right]_0^L = c \rho A \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \left(\frac{-L}{n\pi} \cos(n\pi) + \frac{L}{n\pi} \cos(0) \right)$$

$$= \boxed{c \rho A \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \left(\frac{L}{n\pi} (1 - \cos(n\pi)) \right) \quad n=1,2,3,\dots}$$

2.3.4.b heat flow $\Rightarrow \phi = k \frac{\partial u}{\partial x} \Rightarrow \phi = \sum_{n=1}^{\infty} B_n \left(\frac{L}{n\pi} \right) \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$

$$\phi(0) = k \sum_{n=1}^{\infty} \frac{B_n L}{n\pi} \cos(0) e^{-\left(\frac{n\pi}{L}\right)^2 kt} = \sum_{n=1}^{\infty} \frac{k B_n L}{n\pi} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \quad n=1,2,3,\dots$$

$$\phi(L) = k \sum_{n=1}^{\infty} \frac{B_n L}{n\pi} \cos(n\pi) e^{-\left(\frac{n\pi}{L}\right)^2 kt} = \sum_{n=1}^{\infty} \frac{k B_n L}{n\pi} e^{-\left(\frac{n\pi}{L}\right)^2 kt} (\cos(n\pi)) \quad n=1,2,3,\dots$$

2.3.4.c Heat energy equals initial heat energy plus the time integral of the flow in of the heat energy at the boundaries. (from the book)

2.3.6 $\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx \quad n \geq 0, m \geq 0$

use identity: $\cos(a)\cos(b) = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$

$$\Rightarrow \frac{1}{2} \int_0^L \cos\left(\frac{n\pi x}{L} + \frac{m\pi x}{L}\right) + \cos\left(\frac{n\pi x}{L} - \frac{m\pi x}{L}\right) dx = \frac{1}{2} \int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right) dx$$

$$\text{let } n \neq m \Rightarrow \frac{1}{2} \left[\frac{L}{(n+m)\pi} \sin\left(\frac{(n+m)\pi x}{L}\right) + \frac{L}{(n-m)\pi} \sin\left(\frac{(n-m)\pi x}{L}\right) \right]_0^L$$

$$= \frac{1}{2} \left(\frac{L}{(n+m)\pi} \sin\left(\frac{(n+m)\pi}{L} \cdot L\right) + \frac{L}{(n-m)\pi} \sin\left(\frac{(n-m)\pi}{L} \cdot L\right) - \left(\frac{L}{(n+m)\pi} \sin(0) + \frac{L}{(n-m)\pi} \sin(0) \right) \right)$$

$$= 0 \quad \text{let } n = m \neq 0 \Rightarrow \frac{1}{2} \int_0^L \cos\left(\frac{2n\pi x}{L}\right) + \cos(0) dx = \frac{1}{2} \left[\frac{L}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) + x \right]_0^L$$

$$= \frac{1}{2} \left(\frac{L}{2n\pi} \sin(2n\pi) + L - \left(\frac{L}{2n\pi} \sin(0) + 0 \right) \right) = \frac{L}{2}$$

$$\text{let } n=m=0 \Rightarrow \frac{1}{2} \int_0^L \cos(0) + \cos(0) dx = \frac{1}{2} \int_0^L 2 dx = x \Big|_0^L = L$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n=m \neq 0 \\ L & n=m=0 \end{cases}$$

2.3.8.a $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \alpha u$ B.C. $u(0,t) = 0$
 $u(L,t) = 0$

equilibrium temperature distribution $\Rightarrow \frac{\partial u}{\partial t} = 0$

$$\Rightarrow k \frac{\partial^2 u}{\partial x^2} = \alpha u \quad \alpha > 0 \Rightarrow u(x,t) = ae^{\sqrt{\frac{\alpha}{k}}x} + be^{-\sqrt{\frac{\alpha}{k}}x}$$

$$u(0,t) = a + b = 0 \Rightarrow b = -a$$

$$u(L,t) = ae^{\sqrt{\frac{\alpha}{k}}L} - ae^{-\sqrt{\frac{\alpha}{k}}L} = 0 \Rightarrow a = 0$$

$$\boxed{u(x,t) = 0}$$

2.3.8.b $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \alpha u$ let $u = g(t)\phi(x)$

$$\Rightarrow \phi \frac{dg}{dt} = kg \frac{d^2\phi}{dx^2} - \alpha g\phi \Rightarrow \frac{1}{kg} \frac{dg}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} - \frac{\alpha}{k} = -\lambda$$

$$\frac{dg}{dt} = -\lambda kg \Rightarrow g = Ae^{-\lambda kt}$$

$$\frac{d^2\phi}{dx^2} = \phi \left(\frac{\alpha}{k} - \lambda \right) \quad \text{guess } \phi = e^{rx} \Rightarrow \phi'' = r^2 e^{rx}$$

$$\Rightarrow r^2 e^{rx} = \left(\frac{\alpha}{k} - \lambda \right) e^{rx}$$

$$\Rightarrow r^2 = \left(\frac{\alpha}{k} - \lambda \right) \text{ or } r^2 = -\left(\lambda - \frac{\alpha}{k} \right)$$

$$\Rightarrow r = \pm i \sqrt{\lambda - \frac{\alpha}{k}}$$

$$u(x,t) = (a \cos(\sqrt{\lambda - \frac{\alpha}{k}}x) + b \sin(\sqrt{\lambda - \frac{\alpha}{k}}x)) Ae^{-\lambda kt}$$

$$u(0,t) = a + 0 = 0 \Rightarrow a = 0$$

$$u(L,t) = b \sin(\sqrt{\lambda - \frac{\alpha}{k}}L) Ae^{-\lambda kt} \Rightarrow \sqrt{\lambda - \frac{\alpha}{k}} = \frac{n\pi}{L} \Rightarrow \left(\lambda - \frac{\alpha}{k} \right) = \left(\frac{n\pi}{L} \right)^2$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L} \right)^2 + \frac{\alpha}{k} \quad n=1,2,3,\dots$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\left(\frac{n\pi}{L}\right)^2 + \frac{\alpha}{k}\right)kt}$$

$$\text{but } e^{-\left(\left(\frac{n\pi}{L}\right)^2 + \frac{\alpha}{k}\right)kt} = e^{-\left(\frac{n\pi}{L}\right)^2 kt} e^{-\frac{\alpha}{k}t}$$

$$\boxed{u(x,t) = e^{-\frac{\alpha}{k}t} \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}}$$

$$\text{as } t \rightarrow \infty \quad u \rightarrow 0$$

$$2.4.1.a \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad 0 < x < L, \quad t > 0 \quad \text{B.C.} \quad \frac{\partial u}{\partial x}(0, t) = 0$$

$$\text{I.C.} \quad u(x, 0) = \begin{cases} 0 & x < \frac{L}{2} \\ 1 & x > \frac{L}{2} \end{cases} \quad \frac{\partial u}{\partial x}(L, t) = 0$$

$$\text{let } u(x, t) = (a \cos(\sqrt{\lambda} x) + b \sin(\sqrt{\lambda} x)) A e^{-\lambda k t}$$

$$\frac{\partial u}{\partial x} = (-\sqrt{\lambda} a \sin(\sqrt{\lambda} x) + \sqrt{\lambda} b \cos(\sqrt{\lambda} x)) A e^{-\lambda k t}$$

$$\frac{\partial u}{\partial x}(0, t) = 0 + \sqrt{\lambda} b A e^{-\lambda k t} = 0 \Rightarrow b = 0$$

$$\frac{\partial u}{\partial x}(L, t) = -\sqrt{\lambda} a \sin(\sqrt{\lambda} L) A e^{-\lambda k t} \Rightarrow \sin(\sqrt{\lambda} L) = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad n=0, 1, 2, \dots$$

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t} = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t}$$

$$A_0 = \frac{1}{L} \int_0^{L/2} 0 \, dx + \int_{L/2}^L 1 \, dx = \frac{1}{L} \left[0 \right]_0^{L/2} + \frac{1}{L} \left[x \right]_{L/2}^L = \left(\frac{L}{L} \right) + \left(1 - \frac{1}{2} \right) = \frac{1}{2}$$

$$A_n = \frac{2}{L} \int_0^{L/2} 0 \cdot \cos\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L} \int_{L/2}^L \cos\left(\frac{n\pi x}{L}\right) dx = 0 + \frac{2}{L} \left[\frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \right]_{L/2}^L$$

$$u(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t} = \frac{2}{L} \left(\frac{L}{n\pi} \sin(n\pi) - \frac{L}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right)$$

$$= -\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$2.4.1.b \quad A_0 = \frac{1}{L} \int_0^L (6 + 4 \cos\left(\frac{3\pi x}{L}\right)) dx = \frac{1}{L} \left[6x + \frac{4L}{3\pi} \sin\left(\frac{3\pi x}{L}\right) \right]_0^L = 6 + \frac{4}{3\pi} \sin(3\pi) = 6$$

$$A_n = \frac{2}{L} \int_0^L (6 + 4 \cos\left(\frac{3\pi x}{L}\right)) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L 6 \cos\left(\frac{n\pi x}{L}\right) + 4 \cos\left(\frac{3\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \left(\left[\frac{6L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \right]_0^L + \left\{ \begin{array}{l} \frac{4L}{2} \quad n=3 \\ 0 \quad n \neq 3 \end{array} \right\} \right) = \frac{2}{L} \left(0 + \left\{ \begin{array}{l} 2L \quad n=3 \\ 0 \quad n \neq 3 \end{array} \right\} \right) = \left\{ \begin{array}{l} 4 \quad n=3 \\ 0 \quad n \neq 3 \end{array} \right.$$

$$A_0 = 6 \quad A_3 = 4 \quad \text{all other } A_n = 0$$

2.4.2

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{B.C. } \frac{\partial u}{\partial x}(0, t) = 0 \quad \text{I.C. } u(x, 0) = f(x)$$

$$u(L, t) = 0$$

$$u(x, t) = (a \cos(\sqrt{\lambda} x) + b \sin(\sqrt{\lambda} x)) A e^{-\lambda k t}$$

$$\frac{\partial u}{\partial x} = (-\sqrt{\lambda} a \sin(\sqrt{\lambda} x) + \sqrt{\lambda} b \cos(\sqrt{\lambda} x)) A e^{-\lambda k t}$$

$$u_x(0, t) = (0 + \sqrt{\lambda} b \cos(\sqrt{\lambda} x)) A e^{-\lambda k t} \Rightarrow \cos(\sqrt{\lambda} x) = 0$$

$$\Rightarrow \lambda = \left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2$$

$$u(x, t) = \left(a \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) + b \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) \right) A e^{-\lambda k t}$$

$$u(L, t) = \left(a \cos\left(\frac{(n-\frac{1}{2})\pi L}{L}\right) + b \sin\left(\frac{(n-\frac{1}{2})\pi L}{L}\right) \right) A e^{-\lambda k t} \Rightarrow b = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) e^{-\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2 k t} \quad \text{where } A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) dx$$

2.4.3

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi \quad \phi(0) = \phi(2\pi) \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi)$$

$$\phi = a \cos \sqrt{\lambda} x + b \sin \sqrt{\lambda} x$$

$$\phi' = -\sqrt{\lambda} a \sin \sqrt{\lambda} x + \sqrt{\lambda} b \cos \sqrt{\lambda} x$$

$$\begin{aligned} \phi(0) = a &> \phi(0) = \phi(2\pi) \\ \phi(2\pi) = a &\checkmark \end{aligned}$$

$$\begin{aligned} \phi'(0) = \sqrt{\lambda} b &> \phi'(0) = \phi'(2\pi) \\ \phi'(2\pi) = \sqrt{\lambda} b &\checkmark \end{aligned}$$

this is true for $\sqrt{\lambda} = n$ for all n

$$\Rightarrow \lambda = n^2$$

$$\phi = a \cos(nx) + b \sin(nx) \quad n = 0, 1, 2, \dots$$