Variational Calculus Homework Solutions - Assignment 2

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Please email me at smachr09@evergreen.edu with any corrections or concerns reguarding these solutions.

1. We solve the Brachistochrone problem for a bead moving between $A(0,0)$ and $B(2+3\pi, 2)$. Sketch the solution.

   From lecture, we have that the general solution to the Brachistachrone problem has the parametric form

$$\begin{aligned} x + c &= R(\phi - \sin\phi) \\ y &= R(1 - \cos\phi) \end{aligned}$$

   where $y = 2R\sin^2(\frac{\phi}{2})$. Using point A we find that $0 = 1 - \cos\phi$ which implies that $\phi = 2k\pi$ for some $k \in \mathbf{Z}$. Different values of k simply translate the graph in a manner leaving the trajectory of the bead invarient, so we may as well choose an easy value of k to work with, namely 0. Then $c = 0$ so

$$\begin{aligned} x &= R(\phi - \sin\phi) \\ y &= R(1 - \cos\phi) \end{aligned}$$

   From here we could either use a computer to solve the system for us given that A and B must fall on the solution curve, or we could notice that if we supose that $\phi = \frac{3\pi}{2}$ at point B, our system of equations goes to

$$\begin{aligned} 2 + 3\pi &= R(\tfrac{3\pi}{2} + 1) \\ 2 &= R \end{aligned}$$

   and $R = 2$ satisfies our equation relating ϕ to y. Consequently, this solution works, which we restate

$$\begin{aligned} x &= 2(\phi - \sin\phi) \\ y &= 2(1 - \cos\phi) \end{aligned}$$

2. We solve for the path that light will travel to get from (-1,1) to $(1,1)$ given that the refraction index is given by

$$n(y) = e^y.$$

We use equation 3.1.5 in Lemons to to obtain the equality

$$\begin{aligned} x + c_1 &= \pm \int \frac{dy}{\sqrt{(\frac{e^y}{n^*})^2 - 1}} \\ &= \pm \arctan\left(\frac{\sqrt{e^{2y} - n^2}}{n}\right) + k\pi, \end{aligned}$$

for some $k \in \mathbf{Z}$. We combine terms c_1 and $k\pi$ to get $c = c_1 - k\pi$. Upon doing this, we solve for y, yeilding

$$y = \frac{1}{2}\ln(n^2 \sec^2 |x + c|).$$

Since the y values of our path's initial and terminal points are equal we obtain the equality

$$\sec^2 |-1 + c| = \sec^2 |1 + c|,$$

which simplifies to $|-1+c| = |1+c|$. Since assigning the same sign to to the absolute values makes the c's cancel and leaves us with $2 = 0$, we conclude that we need one of the absolute values must be positive and the other negative. This assumtion readily leads to $c = 0$. We next get rid of the c term in our equation for y above, and our **terminal points** (the initial and final points of our path) to discover that $n = \frac{e}{\sec 1}$. This gives us our final solution for the light's path

$$y = \frac{1}{2}\ln\left(\frac{e^2}{\sec^2 1}\sec^2 |x|\right).$$

3. We find the first integral of the Euler Lagrange equations corresponding to the optimization of the following functional

$$I[y] = \int_{x_1}^{x_2} \frac{1 + (y')^2}{y} dx$$

and solve the resulting differential equation for the terminal points (0,0) and (1,0).

Since the function f inside the integral is independent of x we get the first integral $f - y'\left(\frac{\partial f}{\partial y'}\right) = D$ for some constant D. This gives us the equality

$$\frac{1 + (y')^2}{y} - y'\left(\frac{2y'}{y}\right) = D,$$

which simplifies to

$$y' = \sqrt{1 - Dy}.$$

We use seperation of variables to get the integral equation

$$\int \frac{dy}{\sqrt{1 - Dy}} = x + c,$$

for some constant c. Letting $u = 1 - Dy$ we get the integral

$$-\frac{1}{D}\int u^{-1/2}du = x + c$$

which solves easily, and upon resubstitution and simplification yeilds

$$y = \frac{1}{D}\left(1 - (x+c)^2\frac{D^2}{4}\right)$$

Using our terminal points we obtain the equalities $1 = c^2\frac{D^2}{4}$ and $1 = (1+c)^2\frac{D^2}{4}$, which when solved give us that $D = \pm 4$ and $c = -1/2$, making us done.

4. We solve problems 2.5(b), 3.1 and 3.2 from *Perfect Form* by Lemons.

   2.5 *Variational vs. Direct Method*

   b) Using a first integral of the Euler Lagrange equation, we show that $y = \sin\left(\frac{\pi}{2}x\right)$ stabilizes the integral

   $$I = \int_0^1 \left((y') - \frac{\pi^2 y^2}{4}\right)dx.$$

   Since the equation inside the integral is independent of x, we get the first integral $(y')^2 - \frac{\pi}{4}y^2 - y'(2y') = D_1$ which simplifies to

   $$y' = \sqrt{D_2 - (\frac{\pi}{2})^2 y^2}.$$

   Using seperation of variables and the substitution $u = \frac{\pi}{2}y$ we get the integral

   $$\frac{2}{\pi}\int \frac{du}{\sqrt{D_3 - u^2}} = x + c_1,$$

   which solves out to $\frac{2}{\pi}\arcsin(\frac{u}{\sqrt{D_4}}) = x + c_2$. We substitute $\frac{\pi}{2}y$ for u and solve for y, yeilding

   $$y = D\sin\left(\frac{\pi}{2}x + c\right).$$

   Letting $D = 1$ and $c = 0$ satisfies the equation above for the two terminal ponts, and is thus a valid solution, as desired.

   3.1 We suppose that an atmosphere has index of refraction $n(y) = n_0 - \lambda y$

   a) Show that rays obey the differential equation

   $$\frac{n_0 - \lambda y}{\sqrt{1 + y'^2}} = D$$

   Since $n(y)$ is independent of x, we can use the first integral equation 3.4.5 in Lemons. The desired result follows immediately.

b) We solve the differential equation for initial values $y(0) = 0$ and $y(0) = 0$ using the substitution $n_0 - \lambda y = D\cosh\phi$ to do the integral involved.

We isolate y' in the differential equation found in part a) and get

$$y' = \sqrt{\left(\frac{n_0 - \lambda y}{D}\right)^2 - 1}.$$

Seperating variables we get

$$\int \frac{dy}{\sqrt{\left(\frac{n_0-\lambda y}{D}\right)^2 - 1}} = x + c.$$

Letting $n_0 - \lambda y = D\cosh\phi$, as suggested, we find that $dy = \frac{D\sinh\phi}{\lambda}d\phi$. Making the substitution we get the integral

$$\frac{-D}{\lambda}\int \frac{\sinh\phi}{\sqrt{\cosh^2\phi - 1}}d\phi = x + c.$$

Using the equation $\cosh^2\phi - \sinh^2\phi = 1$ (which follows from the definitions of sinh and cosh in terms of exponentials), we arrive at the equation $-\frac{D}{\lambda}\phi = x + c$ which upon substitution of ϕ in terms of y, and subsequent solution in terms of y, yeilds

$$y = \frac{n_0 - D\cosh(-\lambda x/D + c)}{\lambda},$$

which in turn, upon use of the terminal points gives us the equation

$$y = \frac{n_0}{\lambda}(1 - \cosh(\lambda x/n_0))$$

c) See Figure 1 on page 5 for a computer generated graph for $n_0/\lambda = 1$.

As the graph illustrates, this is concave down. This is indeed what we sould expect, since for the light to get from one point to another, it will want to spend some of it's travel time up at higher elevations (than it would traveling in a straight path) so that it can move faster (than it would traveling a stright path.

3.2 *Designer Rays* We use the generalized Snell's Law (Equation 3.1.3) to design a functional dependence of the index of refraction $n(y)$ which allows for the following rays:

a) $y = ax^2$

From the generalized version of Snell's Law, we get that $n(y) = c\sqrt{1 + y'^2}$. We next note that since $y' = 2ax = 2\sqrt{ay}$ we get

$$n(y) = c\sqrt{1 + 4ay}.$$

b) $y = A\sin(kx)$

We find that $n(y) = c\sqrt{1 + (kA)^2\cos^2(kx)}$. Using the trig identity $\sin^2 x + \cos^2 x = 1$, we are able to rearange in terms of y so that

$$n(y) = c\sqrt{1 + k^2(A^2 - y^2)}.$$

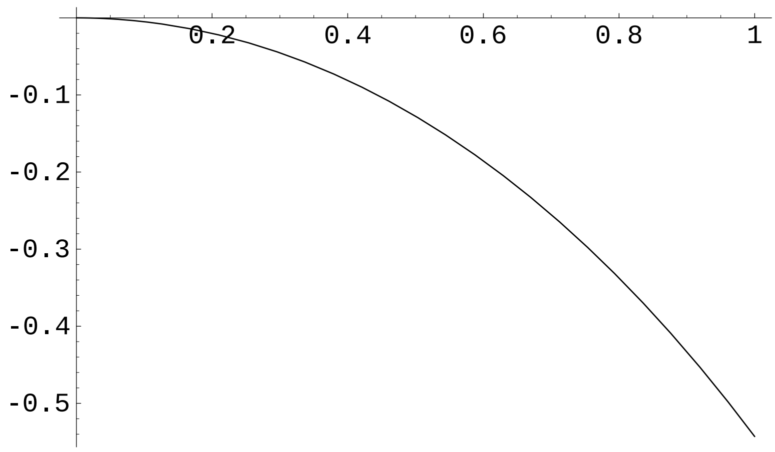

Figure 1: See Problem 3.1 (c)