

Some

### PDE Solutions Week 5 (3.4.3, 3.4.6, 3.4.9, 3.5.4, 4.4.1, 4.4.3)

3.4.3 (a) Let  $f(x) \sim \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$  then for  $n \neq 0$   $a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$\Rightarrow f'(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$  and  $b_n = \frac{2}{L} \int_0^L f'(x) \sin\left(\frac{n\pi x}{L}\right) dx$

using integration by parts let  $v' = f'(x)$   $v = f(x)$

$u = \sin\left(\frac{n\pi x}{L}\right)$   $u' = \cos\left(\frac{n\pi x}{L}\right)$

$\Rightarrow b_n = \frac{2}{L} \left[ f(x) \sin\left(\frac{n\pi x}{L}\right) \right]_0^L - \underbrace{\frac{n\pi}{L} \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}_{a_n}$

Replacing the second term with  $\frac{n\pi a_n}{L}$  and splitting the first term into two pieces to accommodate the jump discontinuity at  $x_0 \Rightarrow$

$b_n = \frac{2}{L} \left( \left[ f(x) \sin\left(\frac{n\pi x}{L}\right) \right]_0^{x_0^-} + \left[ f(x) \sin\left(\frac{n\pi x}{L}\right) \right]_{x_0^+}^L \right) - \frac{n\pi}{L} a_n$   
 $= \frac{2}{L} \left( (\alpha \sin\left(\frac{n\pi x_0}{L}\right) - 0) + (0 - \beta \sin\left(\frac{n\pi x_0}{L}\right)) \right) - \frac{n\pi}{L} a_n$

$b_n = \frac{2}{L} \sin\left(\frac{n\pi x_0}{L}\right) (\alpha - \beta) - \frac{n\pi}{L} a_n$

(b) Let  $f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ , but this series cannot be differentiated term-by-term. Page 121 gives  $a_0$  and  $a_n$  obtained with integration by parts. Expanding these to include  $x_0$  gives:

$a_0 = \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \left( \int_0^{x_0^-} f(x) dx + \int_{x_0^+}^L f(x) dx \right)$   
 $= \frac{1}{L} \left( (f(x_0^-) - f(0)) + (f(L) - f(x_0^+)) \right)$

$a_0 = \frac{1}{L} (\alpha - f(0) + f(L) - \beta)$

Similarly,  $a_n = \frac{2}{L} \left( \int_0^{x_0^-} f(x) \cos\left(\frac{n\pi x}{L}\right) dx + \int_{x_0^+}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \right) + \underbrace{\frac{n\pi}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}_{b_n}$   
 $= \frac{2}{L} \left( \left[ f(x) \cos\left(\frac{n\pi x}{L}\right) \right]_0^{x_0^-} + \left[ f(x) \cos\left(\frac{n\pi x}{L}\right) \right]_{x_0^+}^L + \frac{n\pi}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right)$

continued...

$$\text{so } a_n = \frac{2}{L} \left( (\alpha \cos(\frac{n\pi x_0}{L}) - f(0)) + (f(L) \cos(n\pi) - \beta \cos(\frac{n\pi x_0}{L}) + \frac{n\pi}{L} b_n) \right)$$

$$a_n = \frac{2}{L} \left[ (\alpha - \beta) \cos(\frac{n\pi x_0}{L}) + f(L)(-1)^n - f(0) + \frac{n\pi}{L} b_n \right]$$

3.4.6 The first differentiation is fine since it is always safe to differentiate a cosine series term-by-term. The second differentiation is not allowed since the sine expansion of  $e^x$  has jump discontinuities. Instead use the formula from page 121

$$e^x \sim \frac{1}{L} [e^L - 1] + \sum_{n=1}^{\infty} \left[ \frac{n^2 \pi^2}{L^2} A_n + \frac{2}{L} ((-1)^n e^L - 1) \right] \cos(\frac{n\pi x}{L})$$

We're looking for the form  $e^x \sim A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{L})$

$$\Rightarrow A_0 = \frac{1}{L} [e^L - 1] \quad \text{and} \quad A_n = \frac{n^2 \pi^2}{L^2} A_n + \frac{2}{L} ((-1)^n e^L - 1)$$

$$\text{solving for } A_n \Rightarrow A_n = \frac{2((-1)^n e^L - 1)}{(1 + \frac{n^2 \pi^2}{L^2})} \quad \text{You can check these results with the formulas for } A_0 \text{ and } A_n \text{ on page 121.}$$

$$3.4.9 \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q(x,t) \quad u(0,t) = 0 \quad u(L,t) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(\frac{n\pi x}{L}) \quad \left. \begin{array}{l} u \text{ and } \frac{\partial u}{\partial t} \text{ are continuous} \\ \frac{\partial u}{\partial t} \text{ is piecewise smooth} \end{array} \right\} \Rightarrow \text{Okay to differentiate term-by-term}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right) b_n(t) \cos(\frac{n\pi x}{L}) \Rightarrow \frac{\partial^2 u}{\partial x^2} = - \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 b_n(t) \sin(\frac{n\pi x}{L})$$

$$\text{and } \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{db_n}{dt} \sin(\frac{n\pi x}{L})$$

$$\text{Heat equation } \Rightarrow \sum_{n=1}^{\infty} \frac{db_n}{dt} \sin(\frac{n\pi x}{L}) = -k \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 b_n(t) \sin(\frac{n\pi x}{L}) + q(x,t)$$

$$\text{expand } q(x,t) \text{ such that } q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin(\frac{n\pi x}{L})$$

continued...

$$\text{then } q_n = \frac{2}{L} \int_0^L q(x,t) \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{so } \sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\left(\frac{n\pi x}{L}\right) = -k \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

removing the sums and cancelling the sine terms

$$\Rightarrow \frac{db_n}{dt} = -k \left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n(t)$$

$$\Rightarrow \boxed{\frac{db_n}{dt} = -k \left(\frac{n\pi}{L}\right)^2 b_n(t) + \frac{2}{L} \int_0^L q(x,t) \sin\left(\frac{n\pi x}{L}\right)}$$

an ordinary, linear, first-order diff eq.

3.5.4 (a)  $\cosh(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$

from page 121,

$$f'(x) = \sinh(x) \sim \frac{1}{L} [f(L) - f(0)] + \sum_{n=1}^{\infty} \left[ \frac{n\pi}{L} B_n + \frac{2}{L} ((-1)^n f(L) - f(0)) \right] \cos\left(\frac{n\pi x}{L}\right)$$

but  $\cosh(0) = f(0) = 1$  and  $f(L) = \cosh(L)$

$$f'(x) = \sinh(x) \sim \frac{1}{L} [\cosh(L) - 1] + \sum_{n=1}^{\infty} \left[ \frac{n\pi}{L} B_n + \frac{2}{L} ((-1)^n \cosh(L) - 1) \right] \cos\left(\frac{n\pi x}{L}\right)$$

Differentiating again, this time term-by-term gives

$$f''(x) = \cosh(x) \sim \sum_{n=1}^{\infty} \left[ \left(\frac{n\pi}{L}\right)^2 B_n + \frac{2n\pi}{L^2} ((-1)^n \cosh(L) - 1) \right] \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{Thus } B_n = -\left(\frac{n\pi}{L}\right)^2 B_n + \frac{2n\pi}{L^2} ((-1)^n \cosh(L) - 1)$$

$$\text{Solving for } B_n \text{ gives } \boxed{B_n = \frac{2n\pi}{L^2} ((-1)^n \cosh(L) - 1) / \left(1 + \left(\frac{n\pi}{L}\right)^2\right)}$$

(b)  $\int \cosh(x) dx \sim \int \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) dx$

$$\Rightarrow \sinh(x) \sim A_0 + \sum_{n=1}^{\infty} \frac{-L}{n\pi} B_n \cos\left(\frac{n\pi x}{L}\right), \quad A_0 = \frac{1}{L} \int_0^L \sinh(x) dx = \frac{1}{L} (\cosh(L) - 1)$$

$$\int \sinh(x) dx = \int \frac{1}{L} (\cosh(L) - 1) + \sum_{n=1}^{\infty} \frac{-L}{n\pi} B_n \cos\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow \cosh(x) - 1 = \frac{1}{L} (\cosh(L) - 1) x + \sum_{n=1}^{\infty} -\left(\frac{L}{n\pi}\right)^2 B_n \sin\left(\frac{n\pi x}{L}\right)$$

continued...

$$\Rightarrow \sum_{n=1}^{\infty} B_n \left(1 + \left(\frac{L}{n\pi}\right)^2\right) \sin\left(\frac{n\pi x}{L}\right) = 1 + A_0 x$$

from 3.3.8 and 3.3.12  $B_n \left(1 + \left(\frac{L}{n\pi}\right)^2\right) = \frac{2}{n\pi} \left(1 - (-1)^n\right) + \frac{1}{c} (\cosh(L) - 1) \frac{2L}{n\pi} (-1)^{n+1}$

$$\text{and } B_n = \frac{\frac{2}{n\pi} \left(1 - (-1)^n \cosh(L)\right)}{1 + \left(\frac{L}{n\pi}\right)^2}$$

4.4.1 (a) Natural frequencies are  $c\sqrt{\lambda}$ , but  $\lambda = \left(\frac{n\pi}{L}\right)^2$   
 $\Rightarrow$  natural frequencies are  $\boxed{n\pi c/L \quad n=1,2,3,\dots}$

(b)  $L=H \quad u(0,t)=0 \quad \frac{\partial u}{\partial x}(H,t)=0$

From separation of variables and the wave equation

$$\phi(x) = a \cos \sqrt{\lambda} x + b \sin \sqrt{\lambda} x$$

$$\phi(0) = a = 0 \Rightarrow a = 0$$

$$\phi'(H) = \sqrt{\lambda} b \cos \sqrt{\lambda} H = 0 \Rightarrow \sqrt{\lambda} = \left(\frac{(n-\frac{1}{2})\pi}{H}\right)$$

Since natural frequencies are  $c\sqrt{\lambda}$  we have nat'l freq =  $\boxed{\frac{(n-\frac{1}{2})\pi c}{H} \quad n=1,2,3,\dots}$

(c) let  $H = \frac{L}{2}$ , then  $\left(\frac{(n-\frac{1}{2})\pi c}{H}\right) = \left(\frac{(n-\frac{1}{2})\pi c}{\frac{L}{2}}\right) = \left(\frac{(2n-1)\pi c}{L}\right)$

$$\left(\frac{(2n-1)\pi c}{L}\right) \text{ for } n=1,2,3,\dots \text{ equals } \left(\frac{n\pi c}{L}\right) \text{ for } n=1,3,5,\dots$$

4.4.3 (a)  $\beta > 0$ ,  $\beta$  represents a damping coefficient  
 the vibration is decreasing with respect to time

(b)  $\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$  B.C.  $u(0,t)=0 \quad u(x,0)=f(x)$   
 $u(L,t)=0 \quad \frac{\partial u}{\partial t}(x,0)=g(x)$

let  $u(x,t) = h(t) \phi(x)$

$$\Rightarrow \rho_0 \phi \frac{d^2 h}{dt^2} = T_0 h \frac{d^2 \phi}{dx^2} - \beta \phi \frac{dh}{dt} \Rightarrow \frac{\rho_0}{T_0 h} \frac{d^2 h}{dt^2} + \frac{\beta}{T_0 h} \frac{dh}{dt} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda$$

continued...

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \Rightarrow \phi = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$$

$$\phi(0) = a = 0 \Rightarrow a = 0$$

$$\phi(L) = b\sin\sqrt{\lambda}L = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

$$\rho_0 \frac{d^2h}{dt^2} + \beta \frac{dh}{dt} + \lambda T_0 h = 0 \quad \text{guess } h = e^{st}$$

$$h' = se^{st}$$

$$h'' = s^2 e^{st}$$

$$\rho_0 s^2 + \beta s + \lambda T_0 = 0 \Rightarrow s = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \lambda T_0}}{2\rho_0}$$

assume  $\beta^2 < 4\rho_0^2 T_0 / L^2 \Rightarrow$  we expect complex solutions

$$\text{let } \omega_n = \sqrt{\frac{\beta^2 - 4\rho_0^2 \left(\frac{n\pi}{L}\right)^2 T_0}{4\rho_0^2}} \Rightarrow s = \frac{-\beta \pm i\omega_n}{2\rho_0}$$

$$h = ae^{-\frac{\beta}{2\rho_0}t + i\omega_n t} + be^{-\frac{\beta}{2\rho_0}t - i\omega_n t} \quad \text{or } h = e^{-\beta/2\rho_0 t} (a_n \cos \omega_n t + b_n \sin \omega_n t)$$

$$\text{and } u(x,t) = \sum_{n=1}^{\infty} e^{-\beta/2\rho_0 t} (a_n \cos(\omega_n t) + b_n \sin(\omega_n t)) \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\text{and } \frac{\partial u}{\partial t}(x,0) = g(x) \Rightarrow g(x) = \sum_{n=1}^{\infty} b_n \omega_n \sin\left(\frac{n\pi x}{L}\right) - \frac{\beta}{2\rho_0} \underbrace{\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)}_{f(x)}$$

$$\Rightarrow b_n \omega_n = \frac{\beta a_n}{2\rho_0} + \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$