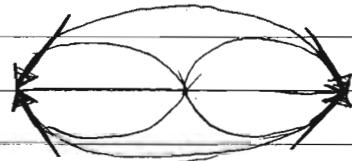


W05 X

PDE Solutions Week 6 (not including 5.5e or 5.8)

$$S.3.2 \quad \rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u + \beta \frac{\partial u}{\partial t}$$

(a) We can identify this as a wave equation with additional terms. Consider a vibrating string



α is a tension force that acts opposite to the string's displacement, $\alpha < 0$

such that for $u < 0$, $\alpha u > 0$ and the string is being forced up to its equilibrium position.

β is a damping coefficient. To be physical, the vertical velocity $\frac{\partial u}{\partial t}$ must decrease as $t \rightarrow \infty$. Therefore $\beta < 0$.

$$(b) \text{ let } u = T(t)\phi(x) \Rightarrow \rho(x)\phi(x) \frac{d^2 T}{dt^2} = T_0 T(t) \frac{d^2 \phi}{dx^2} + \alpha T(t)\phi(x) + \beta(x)\phi(x) \frac{dT}{dt}$$

$$\text{divide by } T\phi\beta \Rightarrow \frac{1}{T\beta(x)} \frac{d^2 T}{dt^2} = T_0 \frac{d^2 \phi}{dx^2} + \frac{\alpha}{\beta} + \frac{1}{T\beta(x)} \frac{dT}{dt}$$

... variables do not separate, 1st and last terms depend on both t and x
let $\beta = c\phi(x)$, divide previous equation by $T\phi c\beta$

$$\Rightarrow \frac{1}{Tc} \frac{d^2 T}{dt^2} = T_0 \frac{d^2 \phi}{dx^2} + \frac{\alpha}{c\beta} + \frac{1}{Tc} \frac{dT}{dt} \quad \begin{matrix} \text{multiply all terms by } c \\ \text{separate variables} \end{matrix}$$

$$\Rightarrow \frac{1}{T} \frac{d^2 T}{dt^2} - \frac{c}{T} \frac{dT}{dt} = T_0 \frac{d^2 \phi}{dx^2} + \frac{\alpha}{c\beta} = -\lambda$$

$$(c) \text{ Space equation: } T_0 \frac{d^2 \phi}{dx^2} + \alpha \phi + \lambda \beta \phi = 0 \quad (\text{after multiplying all terms by } \rho\phi)$$

This is a Sturm-Liouville diff. eq. where $p = T_0 = \text{constant}$

$$q(x) = \alpha(x)$$

$$\sigma(x) = \beta(x)$$

continued...

Time equation: $\frac{d^2T}{dt^2} - c\frac{dT}{dt} + \lambda T = 0$ guess $T = e^{st}$
 $T' = sc e^{st}$
 $T'' = s^2 c e^{st}$

$$\Rightarrow s^2 c e^{st} - c s c e^{st} + \lambda c e^{st} = 0$$

$$s = \frac{c \pm \sqrt{c^2 - 4\lambda}}{2}$$

$$\text{let } d = \sqrt{\frac{c^2 - 4\lambda}{4}}$$

$$T = a e^{\frac{c}{2}+d} + b e^{\frac{c}{2}-d}$$

5.3.3 $\frac{d^2\phi}{dx^2} + \alpha(x) \frac{d\phi}{dx} + [\lambda \beta(x) + \gamma(x)]\phi = 0$

(1) multiply by $H(x)$ $\Rightarrow H(x) \frac{d^2\phi}{dx^2} + H(x) \alpha(x) \frac{d\phi}{dx} + H(x) [\lambda \beta(x) + \gamma(x)]\phi = 0$

(2) We want this in Sturm-Liouville form $\frac{d}{dx} \left(p(x) \frac{d\phi}{dx} \right) + [\lambda \sigma(x) + q(x)]\phi = 0$

Use the product rule to expand the first term of the S.L. diff. eq.

$$\Rightarrow \frac{d}{dx} \left(p(x) \frac{d\phi}{dx} \right) = \frac{dp}{dx} \frac{d\phi}{dx} + p(x) \frac{d^2\phi}{dx^2}$$

By matching coefficients with equation (1), $p(x) = H(x)$

$$\text{and } \frac{dp}{dx} = H(x)\alpha(x)$$

Using separation of variables:

$$\frac{1}{H} dH = \alpha dx \Rightarrow \int \frac{1}{H} dH = \int \alpha dx$$

$$\text{so } \frac{dH}{dx} = H\alpha$$

$$\Rightarrow \ln |H| = \int \alpha dx \Rightarrow H = e^{\int \alpha dx}$$

Additionally, by comparing (1) and (2) $\Rightarrow q(x) = H(x)\gamma(x)$
 $\text{and } \sigma(x) = H(x)\beta(x)$

Put it all together \Rightarrow

$$\frac{d}{dx} \left(e^{\int \alpha dx} \frac{d\phi}{dx} \right) + [\lambda \beta(x) + \gamma(x)]\phi e^{\int \alpha dx} = 0$$

$$5.3.6 \quad \frac{d^2\phi}{dx^2} + \lambda \phi = 0 \quad \frac{d\phi}{dx}(0) = 0 \quad \phi(L) = 0$$

$$(a) \text{ as usual, } \phi = a \cos \sqrt{\lambda} x + b \sin \sqrt{\lambda} x$$

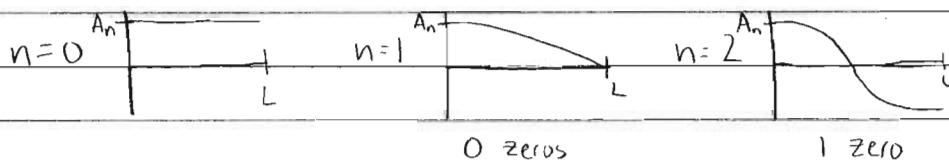
$$\frac{d\phi}{dx}(0) = -\sqrt{\lambda} a \sin(0) + \sqrt{\lambda} b \cos(0) = 0 + \sqrt{\lambda} b \Rightarrow b = 0$$

$$\phi(L) = a \cos(\sqrt{\lambda} L) = 0 \Rightarrow \sqrt{\lambda} = \left(\frac{(n-\frac{1}{2})\pi}{L}\right) \text{ or } \lambda = 0 \text{ since we have a cosine series}$$

$$\text{This means } \lambda_n = \left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2 \text{ where } \lambda_0 = 0, \lambda_1 = \left(\frac{\pi}{2L}\right)^2, n \rightarrow \infty$$

λ_0 is the smallest eigenvalue, there is no largest value

(b) Eigenfunctions are $\phi_n = A_n \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$, where A_n determines the amplitude



etc.
⇒ The n^{th} eigenfunction has $n-1$ zeros

Not proven, but implied

(c) By S.L. theorem 5, $\int_a^b \phi_n(x) \phi_m(x) \sigma(x) dx = 0$ if $\lambda_n \neq \lambda_m$
 $\sigma(x) = 1$ for this diff. eq. and $\phi_n(x) = A_n \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$, $\phi_m(x) = A_m \cos\left(\frac{(m-\frac{1}{2})\pi x}{L}\right)$

$$\text{so } \int_0^L A_n A_m \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) \cos\left(\frac{(m-\frac{1}{2})\pi x}{L}\right) dx = 0 \quad n \neq m$$

(see problem 2.3.6 for worked integral)

This shows that eigenfunctions are orthogonal!

(d) $p = 1$, $\sigma = 1$, $q = 0$ and $d\phi(0) = 0$ and $\phi(L) = 0$

$$\text{Rayleigh quotient: } \lambda = \left[-p \phi \frac{d\phi}{dx} \right]_a^b + \int_a^b [p (\frac{d\phi}{dx})^2 - q \phi^2] dx$$

$$\int_a^b \phi^2 \sigma dx$$

continued...

$$\text{Substituting } \Rightarrow \lambda = \frac{\left[-\phi \frac{d\phi}{dx} \right]_0^L + \int_0^L \left(\frac{d\phi}{dx} \right)^2 dx}{\int_0^L \phi^2 dx}$$

$$\Rightarrow \lambda = \frac{\left(0 \cdot \frac{d\phi}{dx} + \phi \cdot 0 \right) + \int_0^L \left(\frac{d\phi}{dx} \right)^2 dx}{\int_0^L \phi^2 dx}$$

$$\Rightarrow \lambda = \frac{\int_0^L \left(\frac{d\phi}{dx} \right)^2 dx}{\int_0^L \phi^2 dx} \Rightarrow \lambda \geq 0$$

λ cannot be less than 0

$$5.4.1 \quad c\int \frac{\partial u}{\partial t} = \int \frac{\partial}{\partial x} \left(k_0 \frac{\partial u}{\partial x} \right) + \alpha u \quad c, f, k_0, \alpha \text{ are functions of } x$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad u(x, 0) = f(x)$$

$$(a) \text{ let } u = \phi(x)T(t) \Rightarrow c\int \phi \frac{\partial T}{\partial t} = T \int \frac{d}{dx} \left(k_0 \frac{d\phi}{dx} \right) + \alpha T \phi$$

$$\text{divide by } c\phi T \Rightarrow \frac{1}{T} \frac{dT}{dt} = \frac{1}{c\phi} \int \frac{d}{dx} \left(k_0 \frac{d\phi}{dx} \right) + \frac{\alpha}{c\phi} = -\lambda$$

Space equation: $\frac{d}{dx} \left(k_0 \frac{d\phi}{dx} \right) + \alpha \phi + \lambda c\phi = 0$ is a Sturm-Liouville diff. eq.

$$p(x) = k_0(x), q(x) = \alpha(x), \sigma(x) = c(x)\rho(x)$$

$$\text{Rayleigh quotient } \Rightarrow \lambda = \frac{\left[-k_0 \phi \frac{d\phi}{dx} \right]_0^L + \int_0^L \left[k_0 \left(\frac{d\phi}{dx} \right)^2 - \alpha \phi^2 \right] dx}{\int_0^L \phi^2 c\rho dx}$$

since c, f, ρ must be greater than zero, consider $\alpha < 0$
if $\alpha < 0$ then $-\alpha \phi^2 > 0$ and λ is always > 0

$$(b) u(x, t) = \sum_{n=1}^{\infty} a_n \phi_n e^{-\lambda_n t} \quad \text{and} \quad u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n$$

using orthogonality gives equation 5.4.13

$$a_n = \frac{\int_0^L f(x) \phi_n(x) c(x) \rho(x) dx}{\int_0^L \phi_n^2(x) c(x) \rho(x) dx}$$

continued...

(c) from part (a) we assume $\alpha < 0 \Rightarrow \lambda > 0$
 then as $t \rightarrow \infty$ $u(x, t) \rightarrow 0$ since $e^{-\lambda n t} \rightarrow 0$

$$5.4.2 \quad c_p \frac{\partial u}{\partial t} = \frac{d}{dx} \left(k_0 \frac{\partial u}{\partial x} \right) \quad c, p, k_0 \text{ are functions of } x$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0, \quad u(x, 0) = f(x)$$

$$\text{let } u = \phi(x)T(t) \Rightarrow c_p \phi \frac{dT}{dt} = T \frac{d}{dx} \left(k_0 \frac{d\phi}{dx} \right) \text{ divide by } T \phi c_p$$

$$\Rightarrow \frac{1}{T} \frac{dT}{dt} = \frac{1}{c_p \phi} \frac{d}{dx} \left(k_0 \frac{d\phi}{dx} \right) = -\lambda$$

$$\frac{dT}{dt} = -\lambda T \Rightarrow T = a e^{-\lambda t}$$

$$\frac{d}{dx} \left(k_0 \frac{d\phi}{dx} \right) + \lambda c_p \phi = 0 \Rightarrow u(x, t) = \sum_{n=1}^{\infty} a_n \phi_n e^{-\lambda n t}$$

$$a_n = \frac{\int_0^L f(x) \phi_n c_p dx}{\int_0^L \phi_n^2 c_p dx} \text{ as before and } \lambda = \sqrt{\int_0^L k_0 \left(\frac{d\phi}{dx} \right)^2 dx} \geq 0$$

If $\lambda = 0$, the space equation is just $\frac{d}{dx} \left(k_0 \frac{d\phi}{dx} \right) = 0$

but both ends are insulated, $\frac{d\phi}{dx}(L) = 0$ and $\frac{d\phi}{dx}(0) = 0 \Rightarrow \phi = \text{constant}$

$$5.4.3 \quad \frac{\partial u}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \quad u(r, 0) = f(r), \quad |u(0, t)| < \infty, \quad u(a, t) = 0$$

$$u = \phi(r)T(t) \Rightarrow \phi \frac{dT}{dt} = \frac{kT}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) \Rightarrow \frac{1}{T} \frac{dT}{dt} = \frac{k}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) = -\lambda$$

$$T = a e^{-\lambda t} \quad \text{and} \quad \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \frac{\lambda \phi r}{k} = 0 \quad p = r, \quad \sigma = \frac{r}{k}$$

$$u(r, t) = \sum_{n=1}^{\infty} a_n \phi_n e^{-\lambda n t} \quad \text{and} \quad a_n = \frac{\int_0^L f(x) \phi_n \frac{r}{k} dr}{\int_0^L \phi_n^2 \frac{r}{k} dr}$$

$$\text{if } k \text{ is constant, } a_n = \frac{\int_0^L f(x) \phi_n r dr}{\int_0^L \phi_n^2 r dr}$$

5.5.1 $p(u \frac{dv}{dx} - v \frac{du}{dx}) \Big|_a^b = 0$ let $u = \phi_n$ and $v = \phi_m$

such that $p(\phi_n \frac{d\phi_m}{dx} - \phi_m \frac{d\phi_n}{dx}) \Big|_a^b = 0$ letting $a=0$ and $b=L$

$$(1) \Rightarrow p \left[(\phi_n(L) \frac{d\phi_m}{dx}(L) - \phi_m(L) \frac{d\phi_n}{dx}(L)) - (\phi_n(0) \frac{d\phi_m}{dx}(0) - \phi_m(0) \frac{d\phi_n}{dx}(0)) \right] = 0$$

$$(b) \frac{d\phi}{dx}(0) = 0 \text{ and } \phi(L) = 0 \Rightarrow p[(0-0) - (0-0)] = 0 \text{ true}$$

$$(c) \frac{d\phi}{dx}(0) = h\phi(0) \text{ or } \frac{d\phi}{dx}(0) = h\phi(0)$$

$$\text{and } \frac{d\phi}{dx}(L) = 0 \Rightarrow p[(0-0) - (\phi_n h\phi_m - \phi_m h\phi_n)] = 0 \text{ true}$$

$$(e) \phi(a) = \phi(b) \text{ and } \frac{d\phi}{dx}(a) = \frac{d\phi}{dx}(b)$$

let p depend on x

$$\text{then } p(b) \left[\phi_n(b) \frac{d\phi_m}{dx}(b) - \phi_m(b) \frac{d\phi_n}{dx}(b) \right] - p(a) \left[\phi_n(a) \frac{d\phi_m}{dx}(a) - \phi_m(a) \frac{d\phi_n}{dx}(a) \right]$$

$$\Rightarrow p(b) \left[\phi_n(b) \frac{d\phi_m}{dx}(b) - \phi_m(b) \frac{d\phi_n}{dx}(b) \right] - p(a) \left[\phi_n(b) \frac{d\phi_m}{dx}(b) - \phi_m(b) \frac{d\phi_n}{dx}(b) \right] = 0$$

iff $p(b) = p(a)$

5.5.5 $L = \frac{d^2}{dx^2} + 6 \frac{dy}{dx} + 9$

$$(a) L(e^{rx}) = \frac{d^2(e^{rx})}{dx^2} + 6 \frac{d(e^{rx})}{dx} + 9e^{rx} = r^2 e^{rx} + 6r e^{rx} + 9e^{rx}$$

$$= e^{rx}(r^2 + 6r + 9) = e^{rx}(r+3)^2$$

$$(b) L(y) = 0 \Rightarrow \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 9y = 0$$

let $y = e^{rx}$ then $(r+3)^2 e^{rx} = 0 \Rightarrow (r+3)^2 = 0 \Rightarrow r = -3$

$$y = e^{-3x}$$

$$(c) \frac{\partial}{\partial r} L(z(x, r)) = \frac{\partial}{\partial r} \left(\frac{d^2(z(x, r))}{dx^2} + 6 \frac{d(z(x, r))}{dx} + 9(z(x, r)) \right)$$

continued...

$$= \frac{d^2}{dx^2} \left(\frac{\partial z}{\partial r} \right) + 6 \frac{d}{dx} \left(\frac{\partial z}{\partial r} \right) + 9 \left(\frac{\partial z}{\partial r} \right) = L \left(\frac{\partial z}{\partial r} \right)$$

(d) if $z = e^{rx}$, $L \left(\frac{\partial z}{\partial r} \right) = \frac{d^2}{dx^2} \left(\frac{\partial e^{rx}}{\partial r} \right) + 6 \frac{d}{dx} \left(\frac{\partial e^{rx}}{\partial r} \right) + 9 \left(\frac{\partial e^{rx}}{\partial r} \right)$

$$= \frac{d^2}{dx^2} \left(x e^{rx} \right) + 6 \frac{d}{dx} \left(x e^{rx} \right) + 9 x e^{rx}$$

$$\frac{d}{dx} \left(x e^{rx} \right) = e^{rx} + r x e^{rx} \quad \frac{d^2}{dx^2} \left(x e^{rx} \right) = r e^{rx} + r e^{rx} + r^2 x e^{rx}$$

$$\Rightarrow L \left(\frac{\partial (e^{rx})}{\partial r} \right) = e^{rx} (2r + r^2) + 6 e^{rx} (1 + rx) + 9 x e^{rx}$$

$$= \boxed{e^{rx} (2r + r^2 + 6 + 6rx + 9x)}$$

(e) if $y = e^{rx}$, then $e^{rx} (2r + r^2 + 6 + 6rx + 9x) = 0$

Sorry all - can't figure out how to solve this one!