

# PDE Solutions Week 7

S.8.6

assume  $h > 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{wave equation with } \frac{\partial u}{\partial x}(0,t) - hu(0,t) = 0 \quad \underline{\text{B.C.}} \quad u(x,0) = f(x) \quad \underline{\text{I.C.}}$$

$$\text{let } u = \phi(x)g(t)$$

$$\phi \frac{d^2 g}{dt^2} = c^2 g \frac{d^2 \phi}{dx^2}$$

$$\Rightarrow \frac{1}{c^2 g} \frac{d^2 g}{dt^2} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda \quad \text{Time equation: } \frac{d^2 g}{dt^2} = -\lambda c^2 g$$

$$\Rightarrow g = a_1 \cos(c\sqrt{\lambda}t) + b_1 \sin(c\sqrt{\lambda}t)$$

$$\text{Space equation: } \frac{d^2 \phi}{dx^2} = -\lambda \phi$$

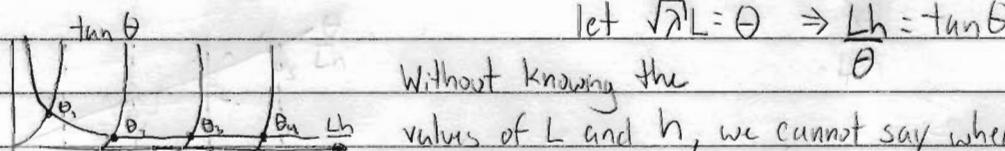
$$\Rightarrow \phi = a_2 \cos(\sqrt{\lambda}x) + b_2 \sin(\sqrt{\lambda}x)$$

$$\text{B.C. } \Rightarrow \phi'(0) - h\phi(0) = 0 \quad \text{and} \quad \phi'(L) = 0$$

$$\phi' = -\sqrt{\lambda}a_2 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}b_2 \cos(\sqrt{\lambda}x)$$

$$\phi'(0) - h\phi(0) = \sqrt{\lambda}b_2 - ha_2 = 0 \Rightarrow a_2 = \frac{\sqrt{\lambda}b_2}{h}$$

$$\phi'(L) = -\frac{\sqrt{\lambda}b_2}{h} \sin(\sqrt{\lambda}L) + \sqrt{\lambda}b_2 \cos(\sqrt{\lambda}L) = 0 \Rightarrow \frac{b_2}{\sqrt{\lambda}} = \tan(\sqrt{\lambda}L)$$



Without knowing the

values of  $L$  and  $h$ , we cannot say where each

$\theta_n$  lies exactly, but we can state a range.

$$0 \leq \theta_1 \leq \frac{\pi}{2}, \quad \pi \leq \theta_2 \leq \frac{3\pi}{2}, \quad 2\pi \leq \theta_3 \leq \frac{5\pi}{2} \text{ etc...}$$

There is an infinite number of  $\theta_n$  as  $n \rightarrow \infty$

$$(a) \sqrt{\lambda}_n = \frac{\theta_n}{L} \quad n=1,2,3\dots \quad \text{and the frequency is given by } c\sqrt{\lambda}_n = \frac{c\theta_n}{L}$$

⇒ infinite different frequencies of oscillation

(b) As  $n \rightarrow \infty$ ,  $\theta_n$  tends to cross the  $\tan \theta$  graph closer and closer to the  $x$ -axis. For large  $n$ ,  $\theta_n \approx (n-1)\pi$

$$(c) u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos\left(\frac{\theta_n x}{L}\right) \cos\left(\frac{c \theta_n t}{L}\right) + B_n \sin\left(\frac{\theta_n x}{L}\right) \sin\left(\frac{c \theta_n t}{L}\right) \right)$$

Apply initial conditions  $f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{\theta_n x}{L}\right)$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \Rightarrow g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{c \theta_n}{L}\right) \sin\left(\frac{\theta_n x}{L}\right) \cos(0)$$

from page 144  $\Rightarrow A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\theta_n x}{L}\right) dx$

$$B_n \left(\frac{c \theta_n}{L}\right) = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{\theta_n x}{L}\right) dx$$

$$5.8.8 \quad \frac{\partial^2 \phi}{\partial x^2} + \lambda \phi = 0 \quad B.C. \quad \phi(0) - \frac{d\phi}{dx}(0) = 0 \quad \phi(1) + \frac{d\phi}{dx}(1) = 0$$

$$(a) p(x) = 1 \quad \sigma(x) = 1 \quad q(x) = 0$$

Rayleigh quotient

$$\Rightarrow \lambda = \frac{\left[ -1 \frac{\phi}{dx} \right]_0^1 + \int_0^1 \left( \frac{d\phi}{dx} \right)^2 dx - 0}{\int_0^1 \phi^2 dx}$$

consider just the first term in the numerator

$$\left[ -1 \frac{\phi}{dx} \right]_0^1 = -\phi(1) \frac{d\phi}{dx}(1) + \phi(0) \frac{d\phi}{dx}(0)$$

but from B.C.  $\phi(0) = \frac{d\phi}{dx}(0)$  and  $\phi(1) = -\frac{d\phi}{dx}(1)$

$$\Rightarrow \lambda = (\phi(1))^2 + (\phi(0))^2 + \int_0^1 \left( \frac{d\phi}{dx} \right)^2 dx$$

all terms are positive, so  $\lambda \geq 0$

$$\text{if } \lambda = 0 \Rightarrow \phi(1) = \phi(0) = \int_0^1 \left( \frac{d\phi}{dx} \right)^2 dx = 0$$

were this the case,  $\frac{\partial^2 \phi}{\partial x^2} + \lambda \phi = 0$  because  $0 + 0 = 0$

this is trivial, therefore  $\lambda > 0$

(b) Let  $L = \frac{d^2}{dx^2}$ , consider eigenfunctions  $\phi_n$  and  $\phi_m$

$$L(\phi_n) + \lambda_n \phi_n = 0 \quad n \neq m, \lambda_n \neq \lambda_m$$

$$L(\phi_m) + \lambda_m \phi_m = 0$$

$$\int_0^1 \phi_m L(\phi_n) + \phi_m \lambda_n \phi_n - (\phi_n L(\phi_m) + \phi_n \lambda_m \phi_m) dx = 0$$

$$\Rightarrow \int_0^1 \phi_m L(\phi_n) - \phi_n L(\phi_m) + \phi_m \phi_n (\lambda_n - \lambda_m) dx = 0$$

$$\Rightarrow \left[ \phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx} \right]_0^1 + \int_0^1 \phi_m \phi_n (\lambda_n - \lambda_m) dx = 0$$

$$\Rightarrow \phi_m(1) \frac{d\phi_n(1)}{dx} - \phi_n(1) \frac{d\phi_m(1)}{dx} - (\phi_m(0) \frac{d\phi_n(0)}{dx} - \phi_n(0) \frac{d\phi_m(0)}{dx}) + \int_0^1 \phi_m \phi_n (\lambda_n - \lambda_m) dx = 0$$

$$\text{from B.C. } \Rightarrow -\phi_m \phi_n + \phi_n \phi_m - \phi_m \phi_n + \phi_n \phi_m + \int_0^1 \phi_m \phi_n (\lambda_n - \lambda_m) dx = 0$$

$$\Rightarrow \int_0^1 \phi_m \phi_n (\lambda_n - \lambda_m) dx = 0 \quad \Rightarrow (\lambda_n - \lambda_m) \int_0^1 \phi_m \phi_n dx = 0$$

$$(\lambda_n - \lambda_m) \neq 0 \Rightarrow \int_0^1 \phi_m \phi_n dx = 0 \Rightarrow \phi_m \text{ and } \phi_n \text{ are orthogonal}$$

$$(c) \frac{d^2\phi}{dx^2} + \lambda \phi = 0 \quad \Rightarrow \quad \begin{aligned} \phi &= a \cos \sqrt{\lambda} x + b \sin \sqrt{\lambda} x \\ \psi &= -\sqrt{\lambda} a \sin \sqrt{\lambda} x + \sqrt{\lambda} b \cos \sqrt{\lambda} x \end{aligned}$$

$$\phi(0) - \frac{d\phi(0)}{dx} = a - (\sqrt{\lambda} b) = 0 \quad \Rightarrow \quad a = \sqrt{\lambda} b$$

$$\phi(1) + \frac{d\phi(1)}{dx} = \sqrt{\lambda} b \cos \sqrt{\lambda} + b \sin \sqrt{\lambda} - \lambda b \sin \sqrt{\lambda} + \sqrt{\lambda} b \cos \sqrt{\lambda} = 0$$

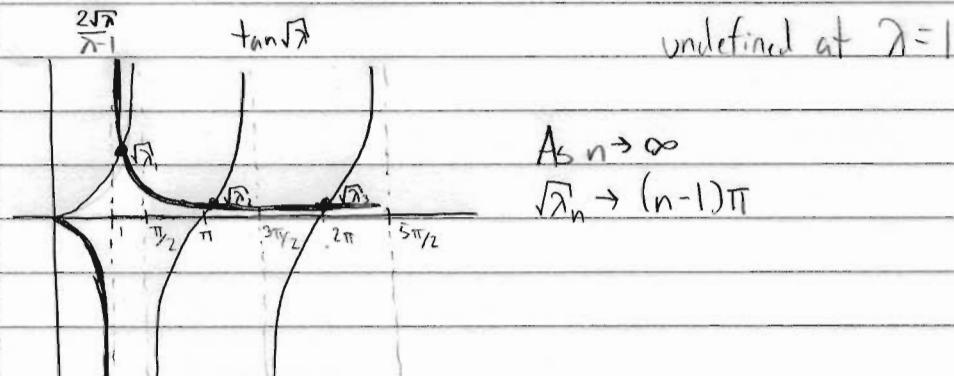
$$\Rightarrow 2\sqrt{\lambda} b \cos \sqrt{\lambda} + (1-\lambda) b \sin \sqrt{\lambda} = 0$$

$$\Rightarrow 2\sqrt{\lambda} \cos \sqrt{\lambda} = (\lambda - 1) \sin \sqrt{\lambda}$$

$$\therefore \Rightarrow \frac{2\sqrt{\lambda}}{\lambda - 1} = \frac{\sin \sqrt{\lambda}}{\cos \sqrt{\lambda}} = \tan \sqrt{\lambda}$$

continued...

$\sqrt{\lambda}$  is our independent variable, so  $\frac{2\sqrt{\lambda}}{\lambda-1}$  will behave like  $\frac{2x}{x^2-1}$



$$\text{As } n \rightarrow \infty \\ \sqrt{\lambda_n} \rightarrow (n-1)\pi$$

$$(d) \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{B.C. } u(0, t) - \frac{\partial u}{\partial x}(0, t) = 0 \quad \text{I.C. } u(x, 0) = f(x)$$

$$u(1, t) + \frac{\partial u}{\partial x}(1, t) = 0$$

$$\text{let } u(x, t) = \phi(x) f(t)$$

$$\Rightarrow \phi \frac{df}{dt} = k f \frac{d^2 \phi}{dx^2} \Rightarrow \frac{1}{kf} \frac{df}{dt} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\gamma$$

$$\text{time equation: } f = e^{-\gamma kt}$$

$$\text{space equation: } \phi = a \cos \sqrt{\lambda} x + b \sin \sqrt{\lambda} x$$

The boundary conditions imply the same B.C. from parts a-c

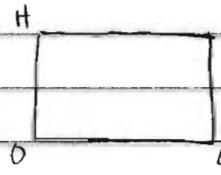
$$\Rightarrow \tan \sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda-1} \Rightarrow \text{eigenfunctions are } \phi_n(x)$$

$$u(x, t) = \sum_{n=1}^{\infty} A_n \phi_n(x) e^{-\gamma kt}$$

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \phi_n(x) = f(x) \Rightarrow A_n = \frac{\int_0^1 f(x) \phi_n(x) dx}{\int_0^1 \phi_n^2(x) dx}$$

$$7.3.1 \quad (a) \quad 0 < x < L, \quad 0 < y < H$$

$$\frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$



$$I.C. \quad u(x, y, 0) = f(x, y)$$

$$B.C. \quad u(0, y, t) = 0 \quad u(L, y, t) = 0 \quad u(x, 0, t) = 0 \quad u(x, H, t) = 0$$

$$\text{let } u = \phi(x, y) f(t) \Rightarrow \phi \frac{df}{dt} = f k \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

$$\Rightarrow \frac{1}{f k} \frac{df}{dt} = \frac{1}{\phi} \nabla^2 \phi = -\lambda \quad \Rightarrow \text{time equation: } f = e^{-\lambda k t}$$

$$\text{let } \phi = g(x) h(y) \Rightarrow h \frac{d^2 g}{dx^2} + g \frac{d^2 h}{dy^2} + \lambda g h = 0$$

$$\Rightarrow \frac{1}{g} \frac{d^2 g}{dx^2} = -\frac{1}{h} \frac{d^2 h}{dy^2} - \lambda = -M \quad \Rightarrow x \text{ equation: } g = a \cos \sqrt{M} x + b \sin \sqrt{M} x$$

$$\text{and } \frac{d^2 h}{dy^2} = (M - \lambda) h \Rightarrow y \text{ equation: } h = c \cos(\sqrt{M-\lambda} y) + d \sin(\sqrt{M-\lambda} y)$$

$$\text{Apply B.C. } u(0, y, t) = g(0) = a = 0 \Rightarrow a = 0$$

$$u(x, 0, t) = h(0) = c = 0 \Rightarrow c = 0$$

$$u(L, y, t) = g(L) = b \sin(\sqrt{M} L) = 0 \Rightarrow \sqrt{M} = \frac{n\pi}{L} \quad n = 1, 2, 3, \dots$$

$$u(x, H, t) = h(H) = d \sin(\sqrt{M-\lambda} H) \Rightarrow \sqrt{M-\lambda} = \frac{m\pi}{H} \quad m = 1, 2, 3, \dots$$

$$\Rightarrow \lambda_{nm} = \left( \frac{m\pi}{H} \right)^2 + \lambda = \left( \frac{m\pi}{H} \right)^2 + \left( \frac{n\pi}{L} \right)^2$$

$$\text{Put it all together} \Rightarrow u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) e^{-\lambda_{nm} k t}$$

$$u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) = f(x, y)$$

$$\text{as usual, } A_{nm} = \frac{\int_0^L \int_0^H f(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dy dx}{\int_0^L \int_0^H \sin^2\left(\frac{n\pi x}{L}\right) \sin^2\left(\frac{m\pi y}{H}\right) dy dx}$$

$$7.3.1 \quad (c) \frac{\partial u}{\partial x}(0, y, t) = 0 \quad \frac{\partial u}{\partial x}(L, y, t) = 0 \quad u(x, 0, t) = 0 \quad u(x, H, t) = 0$$

this is the same problem as (a) but with different boundary conditions

letting  $u(x, y, t) = g(x)h(y)f(t)$  gives:

$$f = e^{-\lambda kt} \quad g = a \cos \sqrt{M}x + b \sin \sqrt{M}x \quad h = c \cos \sqrt{M-\lambda}y + d \sin \sqrt{M-\lambda}y$$

$$\text{Apply B.C. } g' = -\sqrt{M}a \sin \sqrt{M}x + \sqrt{M}b \cos \sqrt{M}x$$

$$\frac{\partial u}{\partial x}(0, y, t) = g'(0) = \sqrt{M}b = 0 \Rightarrow b = 0$$

$$\frac{\partial u}{\partial x}(L, y, t) = g'(L) = -\sqrt{M}a \sin(\sqrt{M}L) = 0 \Rightarrow \sqrt{M} = \frac{n\pi}{L} \quad n=1, 2, 3, \dots$$

$$u(x, 0, t) = h(0) = c = 0 \Rightarrow c = 0$$

$$u(x, H, t) = h(H) = d \sin(\sqrt{M-\lambda}H) = 0 \Rightarrow \sqrt{M-\lambda} = \left(\frac{n\pi}{H}\right)$$

$$\Rightarrow \lambda_{nm} = \left(\frac{n\pi}{H}\right)^2 + M = \left(\frac{n\pi}{H}\right)^2 + \left(\frac{m\pi}{L}\right)^2$$

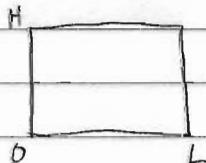
$$\text{Put it all together: } u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) e^{-\lambda_{nm} kt}$$

$$\text{I.C. } u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) = f(x, y)$$

$$\text{and } A_{nm} = \underbrace{\int_0^L \int_0^H f(x, y) \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dy dx}_{\int_0^L \int_0^H \cos^2\left(\frac{n\pi x}{L}\right) \sin^2\left(\frac{m\pi y}{H}\right) dy dx}$$

7.3.4

(b)



$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{I.C. } u(x, y, 0) = 0 \\ \frac{\partial u}{\partial t}(x, y, 0) = f(x, y)$$

$$\text{B.C. } \frac{\partial u}{\partial x}(0, y, t) = 0 \quad \frac{\partial u}{\partial x}(L, y, t) = 0 \quad \frac{\partial u}{\partial y}(x, 0, t) = 0 \quad \frac{\partial u}{\partial y}(x, H, t) = 0$$

$$\text{Let } u = \phi(x, y) f(t) \Rightarrow \frac{1}{f} \frac{d^2 f}{dt^2} = \frac{1}{\phi} \nabla^2 \phi = -\lambda$$

$$\text{time equation: } f = a_1 \cos(c\sqrt{\lambda}t) + b_1 \sin(c\sqrt{\lambda}t)$$

continued...

$$\text{As before, letting } \phi = g(x)h(y) \Rightarrow \frac{1}{g} \frac{d^2g}{dx^2} = -\frac{1}{h} \frac{d^2h}{dy^2} - \lambda = -M$$

$$\Rightarrow x \text{ equation: } g = a_2 \cos \sqrt{M}x + b_2 \sin \sqrt{M}x$$

$$y \text{ equation: } h = a_3 \cos \sqrt{M-\lambda}y + b_3 \sin \sqrt{M-\lambda}y$$

$$\text{Apply B.C. } \frac{\partial u}{\partial x}(0, y, t) = g'(0) = 0 + \sqrt{M}b_2 = 0 \Rightarrow b_2 = 0$$

$$\frac{\partial u}{\partial x}(L, y, t) = g'(L) = -\sqrt{M}a_2 \sin(\sqrt{M}L) = 0 \Rightarrow \sqrt{M} = \frac{n\pi}{L} \quad n=1, 2, 3, \dots$$

$$\frac{\partial u}{\partial y}(x, 0, t) = h'(0) = 0 + \sqrt{M-\lambda}b_3 = 0 \Rightarrow b_3 = 0$$

$$\frac{\partial u}{\partial y}(x, H, t) = h'(H) = -\sqrt{M-\lambda}a_3 \sin(\sqrt{M-\lambda}H) = 0 \Rightarrow \sqrt{M-\lambda} = \frac{m\pi}{H} \quad m=1, 2, 3, \dots$$

$$\Rightarrow \lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

$$\text{Apply I.C. } u(x, y, 0) = f(0) = a_1 = 0 \Rightarrow a_1 = 0$$

$$\text{Put it together: } u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} c \sqrt{\lambda_{nm}} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) \sin(c\sqrt{\lambda_{nm}}t)$$

$$\text{Apply 2nd I.C. } \frac{\partial u}{\partial t}(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} c \sqrt{\lambda_{nm}} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) = f(x, y)$$

$$\text{and } A_{nm} c \sqrt{\lambda_{nm}} = \int_0^L \int_0^H f(x, y) \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) dy dx$$

$$\int_0^L \int_0^H \cos^2\left(\frac{n\pi x}{L}\right) \cos^2\left(\frac{m\pi y}{H}\right) dy dx$$

$$7.3.5 \quad (a) \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - k \frac{\partial u}{\partial t} \quad k > 0$$

This is an equation for a vibrating membrane with damping

$$(b) \text{ let } u(x, y, t) = f(x)g(y)h(t)$$

$$\Rightarrow f_g \frac{d^2h}{dt^2} = c^2 \left( gh \frac{d^2f}{dx^2} + fh \frac{d^2g}{dy^2} \right) - k f g \frac{dh}{dt} \quad \text{divide by } fghc^2$$

$$\Rightarrow \frac{1}{h} \frac{d^2h}{dt^2} = \frac{1}{f} \frac{d^2f}{dx^2} + \frac{1}{g} \frac{d^2g}{dy^2} - \frac{k}{hc^2} \frac{dh}{dt} \Rightarrow \frac{1}{h} \frac{d^2h}{dt^2} + \frac{k}{hc^2} \frac{dh}{dt} = \frac{1}{f} \frac{d^2f}{dx^2} + \frac{1}{g} \frac{d^2g}{dy^2} = -\lambda$$

$$\text{and } \frac{1}{f} \frac{d^2f}{dx^2} = -\frac{1}{g} \frac{d^2g}{dy^2} + \lambda = -M \quad \text{so...}$$

$h'' + kh' = -\lambda hc^2$
$f'' = -Mf$
$g'' = (M-\lambda)g$