

PDE Solutions Week 8

$$7.7.1 \quad \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad u(a, \theta, t) = 0 \quad \frac{\partial u}{\partial t}(r, \theta, 0) = \alpha(r) \sin 3\theta \\ u(r, \theta, 0) = 0$$

From the process of separation of variables done in class and in section 7.7 we know $u(r, \theta, t)$ can be constructed from the parts of the family of product solutions:

$$J_m(\sqrt{\lambda_{mn}} r) \left\{ \begin{array}{l} \cos m\theta \\ \sin m\theta \end{array} \right\} \left\{ \begin{array}{l} \cos c\sqrt{\lambda_{mn}} t \\ \sin c\sqrt{\lambda_{mn}} t \end{array} \right\}$$

$$u(a, \theta, t) = 0 \Rightarrow J_m(\sqrt{\lambda_{mn}} a) = 0 \\ \Rightarrow \sqrt{\lambda_{mn}} a = z_{mn} \Rightarrow \sqrt{\lambda_{mn}} = \frac{z_{mn}}{a}$$

$$u(r, \theta, 0) = 0 \Rightarrow A \cos(0) + B \sin(0) = 0 \Rightarrow A = 0$$

We can eliminate the $\cos c\sqrt{\lambda_{mn}} t$ term from our family of product solutions

$$\text{So } u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\frac{z_{mn}}{a} r\right) \sin\left(\frac{c z_{mn}}{a} t\right) (A_{mn} \cos m\theta + B_{mn} \sin m\theta) \\ \text{let } \frac{c z_{mn}}{a} = \omega_{mn}$$

$$\frac{\partial u}{\partial t}(r, \theta, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} J_m\left(\frac{z_{mn}}{a} r\right) (A_{mn} \cos m\theta + B_{mn} \sin m\theta) = \alpha(r) \sin 3\theta$$

From this initial condition we can eliminate the $\cos m\theta$ term since it doesn't appear on the right-hand-side. We can also conclude that $m=3$ since only $\sin 3\theta$ contributes. All other m 's vanish.

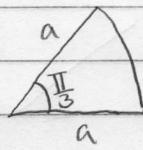
$$\text{So } \sum_{n=1}^{\infty} \omega_{3n} J_3\left(\frac{z_{3n}}{a} r\right) B_{3n} \sin 3\theta = \alpha(r) \sin 3\theta$$

$$\Rightarrow \alpha(r) = \omega_{3n} J_3\left(\frac{z_{3n}}{a} r\right) B_{3n}$$

Using orthogonality with weight r :

$$A_{3n} \omega_{3n} = \int_0^a \alpha(r) J_3(\sqrt{\lambda_{3n}} r) r dr / \int_0^a J_3^2(\sqrt{\lambda_{3n}} r) r dr$$

7.7.4



$$\frac{\partial^2 u}{\partial r^2} = c^2 \nabla^2 u \quad \text{Assume } \lambda > 0$$

Starting from the very beginning: let $u(r, \theta, t) = f(t) \phi(r, \theta)$

$$\phi \frac{d^2 f}{dt^2} = c^2 f \nabla^2 \phi \Rightarrow \frac{1}{f c^2} \frac{d^2 f}{dt^2} = \frac{1}{\phi} \nabla^2 \phi = -\lambda$$

$$\text{time equation: } \frac{d^2 f}{dt^2} = -\lambda f c^2 \Rightarrow f = c_1 \cos(c\sqrt{\lambda}t) + c_2 \sin(c\sqrt{\lambda}t)$$

$$\text{let } \phi = R(r) \Theta(\theta) \text{ and } \nabla^2 \phi + \lambda \phi = 0 = \frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \lambda \phi = 0$$

$$\Rightarrow \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{R}{r^2} \frac{d^2 \Theta}{d\theta^2} + \lambda R \Theta = 0 \Rightarrow \frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} + \lambda r^2 = 0$$

$$\text{angular equation: } \frac{d^2 \Theta}{d\theta^2} = -\lambda \Theta \quad \text{let } \lambda = m^2 \Rightarrow \Theta = a \cos(m\theta) + b \sin(m\theta)$$

$$\text{radial equation: } \frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \lambda r^2 = m^2 \Rightarrow r \frac{d}{dr} \left(r \frac{dR}{dr} \right) + (\lambda r^2 - m^2) R = 0$$

This equation implies $R = J_m(\sqrt{\lambda_{mn}}r)$ and $\Theta_m(\sqrt{\lambda_{mn}}r)$ (Bessel functions)

But we have the restriction that $u(0, \theta, t) < \infty$ and Θ_m blows up at $r=0$
 $\Rightarrow R = J_m(\sqrt{\lambda_{mn}}r)$

$$\text{General Solution: } u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r) (a \cos(m\theta) + b \sin(m\theta)) (c_1 \cos(c\sqrt{\lambda_{mn}}t) + c_2 \sin(c\sqrt{\lambda_{mn}}t))$$

$$(a) \quad (1) u(r, 0, t) = 0 \quad (2) u(r, \frac{\pi}{3}, t) = 0 \quad (3) \frac{\partial u}{\partial r}(a, \theta, t) = 0$$

$$(1) \Rightarrow a \cos(0) + b \sin(0) = 0 \Rightarrow a = 0$$

$$(2) \Rightarrow b \sin\left(\frac{m\pi}{3}\right) = 0 \Rightarrow m = 3\bar{m}, \bar{m} = 0, 1, 2, \dots$$

$$(3) \Rightarrow \sqrt{\lambda_{mn}} J'_m(\sqrt{\lambda_{mn}}a) = 0 \Rightarrow \phi_{mn} = \begin{cases} 1 & m=0, n=1 \\ J_m(\sqrt{\lambda_{mn}}r) & \text{otherwise} \end{cases}$$

where ϕ_{mn} is the infinite set of eigenfunctions

$$\text{so... } \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_n J_m(\sqrt{\lambda_{mn}} r) \sin(3m\theta) \cos(c\sqrt{\lambda_{mn}} t) + B_n J_m(\sqrt{\lambda_{mn}} r) \sin(3m\theta) \sin(c\sqrt{\lambda_{mn}} t)$$

$$(b) (1) u(r, 0, t) = 0 \quad (2) u(r, \frac{\pi}{3}, t) = 0 \quad (3) u(a, \theta, t) = 0$$

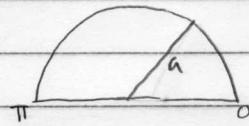
as before, (1) $\Rightarrow a = 0$ and (2) $\Rightarrow m = 3\bar{m}$, $\bar{m} = 0, 1, 2, \dots$

$$(3) \Rightarrow J_m(\sqrt{\lambda_{mn}} a) = 0 \Rightarrow \sqrt{\lambda_{mn}} a = z_{mn} \Rightarrow \sqrt{\lambda_{mn}} = \frac{z_{mn}}{a}$$

$$\text{so... } \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_n J_m\left(\frac{z_{mn}r}{a}\right) \sin(3m\theta) \cos(\omega_{mn}t) + B_n J_m\left(\frac{z_{mn}r}{a}\right) \sin(3m\theta) \sin(\omega_{mn}t)$$

$$\text{let } c\sqrt{\lambda_{mn}} = \omega_{mn}$$

$$7.7.9 (b) \frac{\partial u}{\partial t} = k \nabla^2 u \quad u(r, \theta, 0) = f(r, \theta)$$



$$(1) \frac{\partial u}{\partial \theta}(r, 0, t) = 0$$

$$(2) \frac{\partial u}{\partial \theta}(r, \pi, t) = 0$$

As always, the time equation for the heat equation is $f = Ce^{-\lambda kt}$

And similar to the last two exercises, the angular and radial equations are

$$(4) = a \cos(m\theta) + b \sin(m\theta) \quad \text{and} \quad R = J_m(\sqrt{\lambda_{mn}} r)$$

$$(1) \Rightarrow -ma \sin(0) + mb \cos(0) = 0 \Rightarrow b = 0$$

$$(2) \Rightarrow -ma \sin(m\pi) = 0 \Rightarrow m = \bar{m}, \bar{m} = 1, 2, 3, \dots$$

$$(3) \Rightarrow \phi_{mn}(r) = \begin{cases} 1 & m=0, n=1 \\ J_m(\sqrt{\lambda_{mn}} r) & \text{otherwise} \end{cases}$$

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}} r) \cos(m\theta) e^{-\lambda_{mn} kt}$$

$$\text{As } t \rightarrow \infty \quad e^{-\lambda_{mn} kt} \rightarrow 0$$

$$\text{except } \lambda_{01} = 0 \quad (m=0, n=1)$$

$$\text{therefore } u(r, \theta, t) \rightarrow A_{01} \text{ as } t \rightarrow \infty$$

$$\text{Applying the initial condition } u(r, \theta, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}} r) \cos(m\theta) = f(r, \theta)$$

$$\Rightarrow A_{mn} = \frac{\int_0^\pi \int_0^a f(r, \theta) J_m(\sqrt{\lambda_{mn}} r) \cos(m\theta) r dr d\theta}{\int_0^\pi \int_0^a J_m^2(\sqrt{\lambda_{mn}} r) \cos^2(m\theta) r dr d\theta}$$

$$7.7.10 \quad \frac{\partial u}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \quad \begin{aligned} & \text{(1)} \quad u(a, t) = 0 \\ & \text{(2)} \quad u(r, 0) = f(r) \end{aligned}$$

let $u = f(t)R(r)$

$$\Rightarrow R \frac{df}{dt} = \frac{k f}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) \Rightarrow \frac{1}{k f} \frac{df}{dt} = \frac{1}{r R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = -\lambda$$

time equation: $f = e^{-\lambda k t}$

$$\text{radial equation: } \frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \lambda R = 0 \quad \text{multiplying by } r^2$$

$$\Rightarrow r \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \lambda R r^2 = 0 \quad \text{which we recognize from previous problems (without the usual } m^2 \text{ term)}$$

$$\Rightarrow R = J_0(\sqrt{\lambda_n} r) \quad \text{note: } m=0, \text{ so only } J_0 \text{ contributes}$$

$$\text{So } u(r, t) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n} r) e^{-\lambda_n k t}$$

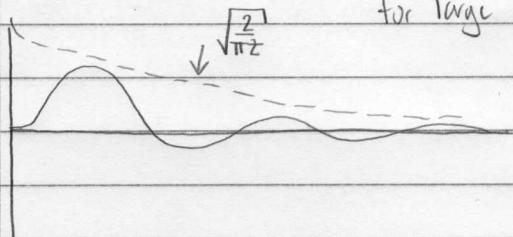
$$(1) \Rightarrow J_0(\sqrt{\lambda_n} a) = 0 \Rightarrow \sqrt{\lambda_n} a = z_n \Rightarrow \sqrt{\lambda_n} = \frac{z_n}{a}$$

$$(2) \Rightarrow \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n} r) = f(r) \Rightarrow a_n = \frac{\int_0^a f(r) J_0(\sqrt{\lambda_n} r) r dr}{\int_0^a J_0^2(\sqrt{\lambda_n} r) r dr}$$

As $t \rightarrow \infty$, $e^{-\lambda_n k t} \rightarrow 0$

$$7.8.5 \quad (a) \text{ Sketch } J_4(z) \quad \text{for small } z \quad J_4(z) \approx \frac{1}{4!} z^4 \quad (\text{using equation 7.8.2})$$

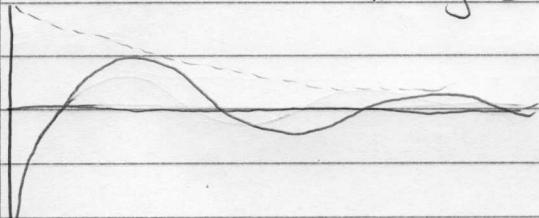
$$\text{for large } z \quad J_4(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - 2\pi\right)$$



$J_4(z)$ leaves the origin like z^4 and settles into a decaying oscillation

$$(b) \text{ Sketch } Y_4(z) \quad \text{for small } z \quad Y_4(z) \approx -\frac{2}{\pi} z^{-1}$$

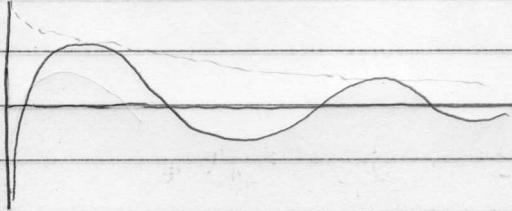
$$\text{for large } z \quad Y_4(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{\pi}{2}\right)$$



(c) Sketch $Y_0(z)$

$$\text{for small } z \quad Y_0(z) \approx \frac{2}{\pi} \ln z$$

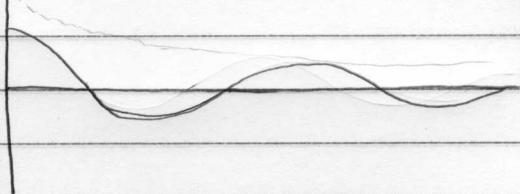
$$\text{for large } z \quad Y_0(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right)$$



(d) Sketch $J_0(z)$

$$\text{for small } z \quad J_0(z) \approx 1$$

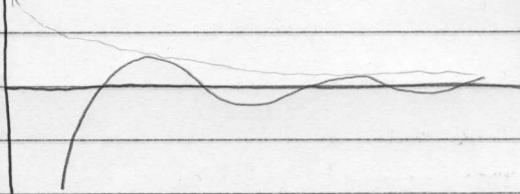
$$\text{for large } z \quad J_0(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right)$$



(e) Sketch $Y_5(z)$

$$\text{for small } z \quad Y_5(z) \approx -\frac{768}{\pi} z^{-5}$$

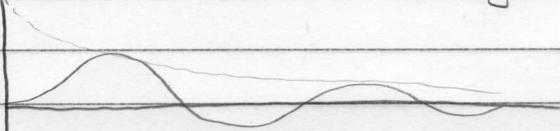
$$\text{for large } z \quad Y_5(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{5\pi}{2}\right)$$



(f) Sketch $J_2(z)$

$$\text{for small } z \quad J_2(z) \approx \frac{1}{8} z^2$$

$$\text{for large } z \quad J_2(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \pi\right)$$



7.8.6

The eigenfunctions for a vibrating circular membrane are $J_m\left(\frac{z_{mn}}{a}\right)$
where $\frac{z_{mn}}{a}$ is the frequency

$$\text{for large } z, J_n(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$$

knowing what we know about cosines, zeroes occur in this approximation when

$$z - \frac{\pi}{4} = -\frac{\pi}{2} + s\pi \quad \text{where } s \text{ is some integer}$$

$$\text{or } z = \pi\left(\frac{1}{4} - \frac{1}{2} + s\right) = \pi\left(s - \frac{1}{4}\right)$$

so $\frac{z_{mn}}{a} = \pi\left(s - \frac{1}{4}\right)$ are the natural frequencies