This test is due on Monday, Feb 12th at 1:00. You may refer to your notes and textbook, but you must not consult with other people.

1. Write down and simplify first integrals of the Euler-Lagrange equations corresponding to the optimization the following functionals

   (a) $\int_a^b \frac{\sqrt{1+y'^2}}{1+y} dx$

   (b) $\int_\alpha^\beta \sqrt{r^2 r'^2 + r^4} d\theta$

2. An object follows a path parameterized in plane polar coordinates by $r(t)$ and $\theta(t)$ which minimizes the following functional.

$$\int \left(\tfrac{1}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{r} \right) dt$$

   Write down Euler lagrange equations for the two dependent variables, and then by finding the first integrals, find a system of first order differential equations describing the motion of the object.

3. On a hot day you can sometimes see a mirage, which is due to light bending upward as it approaches the hot surfaces of the road. Assuming that the refractive index varies linearly with height above the surface in the following way,

$$n = 1 + \alpha y$$

   with α a small positive number. Use Fermat's principle to derive a differential equation describing the path of light. Find the general solution for the path of light rays and describe the type of path the rays follow.

4. The electrostatic potential at the origin due to a uniformly charged wire with polar equation $r(\theta)$ connecting two points A and B is given by the expression.

$$V[r] = k \int_A^B \frac{1}{r} ds,$$

   were k is a constant, r is the distance from a the origin to each point on the wire and ds is the arclength element along the wire. By expressing ds in polar coordinates, find the equation of the wire that minimizes the potential at the origin. Sketch this curve and describe it.