

1. Compute the following double integrals

(a)

$$I = \int \int_{[1,2] \times [0,1]} x \sqrt{1+y+x^2} dA$$

In this case the order you choose for integration won't make a big difference. Let's do the x integration first

$$I = \int_0^1 \int_1^2 x \sqrt{1+y+x^2} dx dy$$

Now we need to evaluate $\int x \sqrt{1+y+x^2} dx$. We'll do this by substitution. Let $u = 1 + x^2 + y$ then, since we keep y constant. $du = 2x dx \Rightarrow dx = du/2x$. Thus

$$\int x \sqrt{1+y+x^2} dx = \int x \sqrt{u} du / 2x = \frac{1}{2} \int \sqrt{u} du = \frac{1}{2} \frac{u^{3/2}}{\frac{3}{2}} = \frac{u^{3/2}}{3} = \frac{(1+y+x^2)^{3/2}}{3}.$$

So our double integral reduces to the single integral

$$I = \frac{1}{3} \int_0^1 \left[(1+y+x^2)^{3/2} \right]_1^2 dy = \frac{1}{3} \int_0^1 (5+y)^{3/2} - (2+y)^{3/2} dy$$

which then becomes

$$I = \frac{1}{3} \left[\frac{(5+y)^{5/2} - (2+y)^{5/2}}{\frac{5}{2}} \right]_0^1 = \frac{2}{15} \left((6)^{5/2} - (3)^{5/2} - (5)^{5/2} + (2)^{5/2} \right)$$

(b)

$$\int \int_{[0,1] \times [1,2]} \frac{x}{x+y} dA$$

The y integral looks easiest so let's do that first

$$I = \int_0^1 \int_1^2 \frac{x}{x+y} dy dx = \int_0^1 [x \ln |x+y|]_1^2 dx = \int_0^1 x \ln(2+x) - x \ln(1+x) dx$$

Now we are left with single integrals of the form $\int x \ln(x+a)$, which we can either look up in a table, do with technology, or do by parts. I get

$$\int x \ln(x+a) = \frac{ax}{2} - \frac{x^2}{4} - \frac{1}{2} a^2 \ln(x+a) + \frac{1}{2} x^2 \ln(x+a)$$

so that

$$\begin{aligned} I &= \left[\frac{2x}{2} - \frac{x^2}{4} - \frac{1}{2} 2^2 \ln(x+2) + \frac{1}{2} x^2 \ln(x+2) - \left(\frac{x}{2} - \frac{x^2}{4} - \frac{1}{2} \ln(x+1) + \frac{1}{2} x^2 \ln(x+1) \right) \right]_0^1 \\ &= \frac{1}{2} - \frac{3}{2} \ln 3 + 2 \ln 2 \end{aligned}$$

(c)

$$\iint_{[0,1] \times [1,2]} ye^{xy} dA$$

It looks like it might be best to do the x integral first

$$\begin{aligned} I &= \int_1^2 \int_0^1 ye^{xy} dy dx = \int_0^1 \left[y \frac{e^{xy}}{y} \right]_0^1 dy = \int_1^2 [e^{xy}]_0^1 dy \\ &= \int_1^2 (e^y - 1) dy = [e^y - y]_1^2 = (e^2 - 2 - e + 1) = e^2 - e - 1 \end{aligned}$$