Thermal Physics

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The probability that the system is in the state s of energy $s\hbar\omega$ is given by the Boltzmann factor:

$$P(s) = \frac{\exp(-s\hbar\omega/\tau)}{Z}. \tag{4}$$

The thermal average value of s is   s = # of photons in mode of

$$\langle s \rangle = \sum_{s=0}^{\infty} sP(s) = Z^{-1}\sum s\exp(-s\hbar\omega/\tau). \tag{5}$$

With $y \equiv \hbar\omega/\tau$, the summation on the right-hand side has the form:

$$\sum s\exp(-sy) = -\frac{d}{dy}\sum\exp(-sy)$$

$$= -\frac{d}{dy}\left(\frac{1}{1-\exp(-y)}\right) = \frac{\exp(-y)}{[1-\exp(-y)]^2}.$$

From (3) and (5) we find

$$\langle s \rangle = \frac{\exp(-y)}{1-\exp(-y)},$$

or

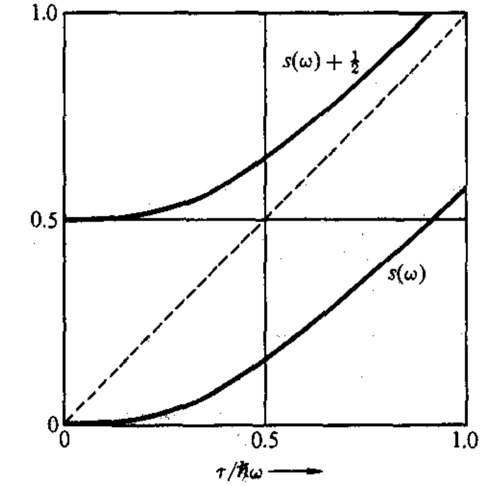
$$\boxed{\langle s \rangle = \frac{1}{\exp(\hbar\omega/\tau) - 1}.} \tag{6}$$

This is the **Planck distribution function** for the thermal average number of photons (Figure 4.3) in a single mode of frequency ω. Equally, it is the average number of phonons in the mode. The result applies to any kind of wave field with energy in the form of (1).

PLANCK LAW AND STEFAN-BOLTZMANN LAW

The thermal average energy in the mode is

$$\langle \varepsilon \rangle = \langle s \rangle \hbar\omega = \frac{\hbar\omega}{\exp(\hbar\omega/\tau) - 1}. \tag{7}$$



distribution as a function
perature $\tau/\hbar\omega$. Here $\langle s(\omega)\rangle$
age of the number of
e of frequency ω. A plot of
given, where $\frac{1}{2}$ is the effective
cy of the mode; the dashed
asymptote. Note that we

$\frac{1}{2} = \frac{1}{2}\coth(\hbar\omega/2\tau)$.

The high temperature limit $\tau \gg \hbar\omega$ is often called the classical limit. Here $\exp(\hbar\omega/\tau)$ may be approximated as $1 + \hbar\omega/\tau + \cdots$, whence the classical average energy is

$$\langle \varepsilon \rangle \simeq \tau. \tag{8}$$

$E_{\parallel}$ walls $= 0$

There is an infinite number of electromagnetic modes within any cavity. Each mode n has its own frequency ω_n. For radiation confined within a perfectly conducting cavity in the form of a cube of edge L, there is a set of modes of the form

vanishes at $y = 0, L$ and at $z = 0, L$

?

$$E_x = E_{x0} \sin \omega t \cos(n_x \pi x/L) \sin(n_y \pi y/L) \sin(n_z \pi z/L) \ , \tag{9a}$$

$$E_y = E_{y0} \sin \omega t \sin(n_x \pi x/L) \cos(n_y \pi y/L) \sin(n_z \pi z/L) \ , \tag{9b}$$

$$E_z = E_{z0} \sin \omega t \sin(n_x \pi x/L) \sin(n_y \pi y/L) \cos(n_z \pi z/L). \tag{9c}$$

Here E_x, E_y and E_z are the three electric field components, and E_{x0}, E_{y0} and E_{z0} are the corresponding amplitudes. The three components are not independent, because the field must be divergence-free:

NO CHARGE INSIDE

$$\operatorname{div} \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0. \tag{10}$$

When we insert (9) into (10) and drop all common factors, we find the condition

$$E_{x0} n_x + E_{y0} n_y + E_{z0} n_z = \mathbf{E}_0 \cdot \mathbf{n} = 0. \tag{11}$$

This states that the field vectors must be perpendicular to the vector $\mathbf{n}$ with the components n_x, n_y and n_z, so that the electromagnetic field in the cavity is a $\boxed{\text{transversely polarized}}$ field. The polarization direction is defined as the direction of $\mathbf{E}_0$.

For a given triplet n_x, n_y, n_z we can choose two mutually perpendicular polarization directions, so that there are two distinct modes for each triplet n_x, n_y, n_z.

On substitution of (9) in the wave equation $\nabla^2 E = \frac{\partial^2}{c^2 \partial t^2} E$

$$c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)E_z = \frac{\partial^2 E_z}{\partial t^2} \ , \tag{12}$$

with c the velocity of light, we find $\boxed{\omega}$

$$c^2\pi^2(n_x^2 + n_y^2 + n_z^2) = \omega^2 L^2. \tag{13}$$

This determines the frequency ω of the mode in terms of the triplet of integers n_x, n_y, n_z. If we define

not necessarily an integer

$$n \equiv (n_x^2 + n_y^2 + n_z^2)^{1/2} \ , \tag{14}$$

$\neq s = $ # of photons

then the frequencies are of the form determined by BC

$$\omega_n = n\pi c/L. \tag{15}$$

The total energy of the photons in the cavity is, from (7),

energy of each mode

$$U = \sum_n \langle \varepsilon_n \rangle = \sum_n \frac{\hbar\omega_n}{\exp(\hbar\omega_n/\tau) - 1}. \tag{16}$$

all modes $n_x n_y n_z$

The sum is over the triplet of integers n_x, n_y, n_z. Positive integers alone will describe all independent modes of the form (9). We replace the sum over n_x, n_y, n_z by an integral over the volume element $dn_x\, dn_y\, dn_z$ in the space of the mode indices. That is, we set

$$\sum_n (\cdots) = \tfrac{1}{8}\int_0^\infty 4\pi n^2\, dn(\cdots) \ , \tag{17}$$

$4\pi r^2 \, r^2 dr \; r^2 dr \, d\varphi \, \sin\theta d\theta \quad = \int \frac{A \cdot \hat{e}}{4\pi r^2} \rho \, d^3 r$

$u = \frac{A}{4\pi} \int_0^1 \cos\theta \, \rho \, d\cos\theta \; 2\pi \int^{ct} dr = \frac{Ac}{2\pi 2} \, dt \, \rho \rightarrow \frac{u}{A dt} = \frac{c\rho}{4}$

where the factor $\frac{1}{8} = (\frac{1}{2})^3$ arises because only the positive octant of the space is involved. We now multiply the sum or integral by a factor of 2 because there are two independent polarizations of the electromagnetic field (two independent sets of cavity modes). Thus for each frequency

$$U = \pi \int_0^\infty dn \, n^2 \frac{\hbar\omega_n}{\exp(\hbar\omega_n/\tau) - 1} \qquad \omega_c = \frac{n\pi c}{L}$$

$$= (\pi^2 \hbar c/L) \int_0^\infty dn \, n^3 \frac{1}{\exp(\hbar c n \pi / L\tau) - 1}, \tag{18}$$

with (15) for ω_n. Standard practice is to transform the definite integral to one over a dimensionless variable. We set $x \equiv \pi\hbar cn/L\tau$, and (18) becomes

$$U = (\pi^2 \hbar c/L)(\tau L/\pi\hbar c)^4 \int_0^\infty dx \frac{x^3}{\exp x - 1}. \tag{19}$$

The definite integral has the value $\pi^4/15$; it is found in good standard tables such as Dwight (cited in the general references). The energy per unit volume is

$$\boxed{\frac{U}{V} = \frac{\pi^2}{15\hbar^3 c^3} \tau^4}, \tag{20}$$

with the volume $V = L^3$. The result that the radiant energy density is proportional to the fourth power of the temperature is known as the **Stefan-Boltzmann law of radiation**.

For many applications of this theory we decompose (20) into the spectral density of the radiation. The spectral density is defined as the energy per unit volume per unit frequency range, and is denoted as u_ω. We can find u_ω from (18) rewritten in terms of ω: $= \frac{c\pi n}{L}$

$$\rho_\lambda = U/V = \int d\omega \, u_\omega = \frac{\hbar}{\pi^2 c^3} \int d\omega \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}, \tag{21}$$

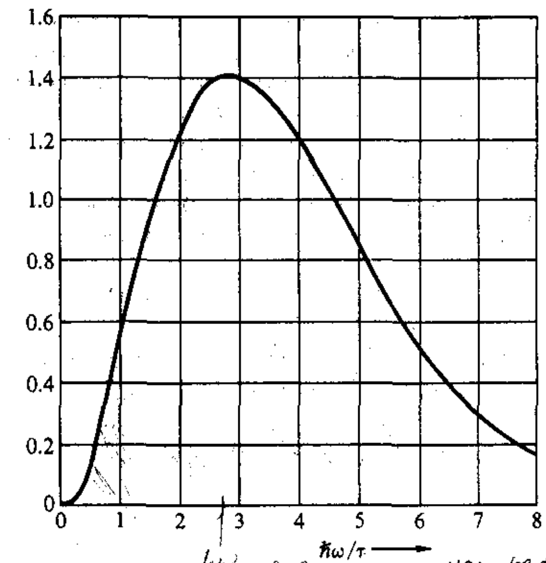
so that the spectral density is

$$\rho = \boxed{u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}}. \qquad \frac{\text{energy}}{\text{volume} \cdot \text{unit } d\omega} \tag{22}$$

$= \frac{\tau^3}{\pi^2 c^3 \hbar^2} \frac{x^3}{e^x - 1}$

$\frac{\tau^3}{\pi^2 c^3 \hbar^2} \frac{\hbar^3\omega^3}{\tau^3}$

energy density – measure of radiation amplitude



area under = total energy over all freqs

τ determines ω for which max radiation

$\frac{\hbar\omega}{\tau} = 2.82 \rightarrow$ max radiation at $\omega' =$

measure ω', FIND τ.

Figure 4.4 Plot of $x^3/(e^x - 1)$ with $x \equiv \hbar\omega/\tau$. This function is involved in the Planck radiation law for the spectral density u_ω. The temperature of a black body may be found from the frequency ω_{max} at which the radiant energy density is a maximum, per unit frequency range. This frequency is directly proportional to the temperature.

This result is the **Planck radiation law**; it gives the frequency distribution of thermal radiation (Figure 4.4). Quantum theory began here.

The entropy of the thermal photons can be found from the relation (3.34a) at constant volume: $d\sigma = dU/\tau$, whence from (20),

$$\int d\sigma = \frac{4\pi^2 V}{15\hbar^3 c^3} \int \tau^2 \, d\tau.$$

Thus the entropy is

$$\int \frac{1}{\tau} \frac{dU}{d\tau} d\tau = \sigma(\tau) = (4\pi^2 V/45)(\tau/\hbar c)^3. \tag{23}$$

The constant of integration is zero, from (3.55) and the relation between F and σ.

$F = -\tau \ln Z \qquad \sigma = -\frac{\partial F}{\partial \tau} = \ln Z$

if $\sigma(\tau = 0) = 0$

A process carried out at constant photon entropy will have $\boxed{V\tau^3 = \text{constant}}$.

The measurement of high temperatures depends on the flux of radiant energy from a small hole in the wall of a cavity maintained at the temperature of interest. Such a hole is said to radiate as a black body—which means that the radiation emission is characteristic of a thermal equilibrium distribution. The energy flux density J_U is defined as the rate of energy emission per unit area. The flux density is of the order of the energy contained in a column of unit area and length equal to the velocity of light times the unit of time. Thus,

$$J_U = [cU(\tau)/V] \times (\text{geometrical factor}). \tag{24}$$

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The geometrical factor is equal to $\frac{1}{4}$; the derivation is the subject of Problem 15. The final result for the radiant energy flux is

$$J_U = \frac{cU(\tau)}{4V} = \frac{\pi^2\tau^4}{60\hbar^3c^2} , \tag{25}$$

by use of (20) for the energy density U/V. The result is often written as

$$J_U = \sigma_B T^4; \tag{26}$$

the **Stefan-Boltzmann constant**

$$\sigma_B \equiv \pi^2 k_B{}^4/60\hbar^3c^2 \tag{26a}$$

has the value 5.670×10^{-8} W m^{-2} K^{-4} or 5.670×10^{-5} erg cm^{-2} s^{-1} K^{-4}. (Here σ_B is not the entropy.) A body that radiates at this rate is said to radiate as a black body. A small hole in a cavity whose walls are in thermal equilibrium at temperature T will radiate as a black body at the rate given in (26). The rate is independent of the physical constitution of the walls of the cavity and depends only on the temperature.

Emission and Absorption: Kirchhoff Law

The ability of a surface to emit radiation is proportional to the ability of the surface to absorb radiation. We demonstrate this relation, first for a black body or black surface and, second, for a surface with arbitrary properties. An object is defined to be black in a given frequency range if all electromagnetic radiation incident upon it in that range is absorbed. By this definition a hole in a cavity is black if the hole is small enough that radiation incident through the hole will

$\tau = -273$ C

radiation pressure $= \frac{F}{\text{Area}} = \frac{dP/dt}{A} = \frac{U}{ct\,A}$

reflect enough times from the cavity walls to be absorbed in the cavity with negligible loss back through the hole.

The radiant energy flux density J_U from a black surface at temperature τ is equal to the radiant energy flux density J_U emitted from a small hole in a cavity at the same temperature. To prove this, let us close the hole with the black surface, hereafter called the object. In thermal equilibrium the thermal average energy flux from the black object to the interior of the cavity must be equal, but opposite, to the thermal average energy flux from the cavity to the black object.

We prove the following: If a non-black object at temperature τ absorbs a fraction a of the radiation incident upon it, the radiation flux emitted by the object will be a times the radiation flux emitted by a black body at the same temperature. Let a denote the **absorptivity** and e the **emissivity**, where the emissivity is defined so that the radiation flux emitted by the object is e times the flux emitted by a black body at the same temperature. The object must emit at the same rate as it absorbs if equilibrium is to be maintained. It follows that $a = e$. This is the **Kirchhoff law**. For the special case of a perfect reflector, a is zero, whence e is zero. A perfect reflector does not radiate. doesn't heat up.

The arguments can be generalized to apply to the radiation at any frequency, as between ω and $\omega + d\omega$. We insert a filter between the object and the hole in the black body. Let the filter reflect perfectly outside this frequency range, and let it transmit perfectly within this range. The flux equality arguments now apply to the transmitted spectral band, so that $a(\omega) = e(\omega)$ for any surface in thermal equilibrium.

Estimation of Surface Temperature

One way to estimate the surface temperature of a hot body such as a star is from the frequency at which the maximum emission of radiant energy takes place (see Figure 4.4). What this frequency is depends on whether we look at the energy flux per unit frequency range or per unit wavelength range. For u_ω, the energy density per unit frequency range, the maximum is given from the Planck law, Eq. (22), as

$$\frac{d}{dx}\left(\frac{x^3}{\exp x - 1}\right) = 0 ,$$

or

$$3 - 3\exp(-x) = x.$$