Blackbody Radiation

Introduction

First radiation process to look at: radiation in thermal equilibrium with itself: blackbody radiation

Assumptions:

1. Photons are Bosons, i.e., more than one photon per phase space cell possible.
2. Photons are in thermodynamic equilibrium at all frequencies.

Outline of computation:

1. Compute mean energy of photons of frequency ν in phase space cell, $\langle E(\nu)\rangle$
2. Compute number of phase space cells as a function of frequency, $N(\nu)$.
3. Compute photon spectrum as product $\langle E(\nu)\rangle \cdot N(\nu)$.

Derivation: Step 1, I

First step: Mean energy of photons of frequency ν in phase space cell.

Describe phase space cell as box $\Longrightarrow$ Photons: $\sim$solution of QM harmonic oscillator $\Longrightarrow$ Total energy of box with n photons:

$$E_n = \left(n + \frac{1}{2}\right) \cdot h\nu \tag{3.1}$$

where $\frac{1}{2}h\nu$: ground state energy (unobservable).

Probability that oscillator is in nth state from Boltzmann:

$$P_n(\nu, T) = \frac{\exp\left(-\left(n+\frac{1}{2}\right)h\nu\right)}{\sum_{n'} \exp\left(-\left(n'+\frac{1}{2}\right)h\nu\right)} = \frac{\exp(-nh\nu/kT)}{\sum_{n'} \exp(-n'h\nu/kT)} \tag{3.2}$$

Therefore, average energy per phase cell:

$$\langle E \rangle = \sum_n E_n P_n(\nu, T) \tag{3.3}$$

$$= \sum_n \left\{ \left(n + \frac{1}{2}\right) h\nu \cdot \frac{\exp(-nh\nu/kT)}{\sum_{n'} \exp(-n'h\nu/kT)} \right\} \tag{3.4}$$

introducing $x = h\nu/kT$

$$= \frac{kT \sum_n \left(n + \frac{1}{2}\right) x \exp(-nx)}{\sum_n \exp(-nx)} \tag{3.5}$$

$$= kT \left\{ \frac{\sum_n nx \exp(-nx)}{\sum_n \exp(-nx)} + \frac{x}{2} \right\} \tag{3.6}$$

Derivation: Step 1, II

To evaluate $\langle E\rangle$, need to compute the geometric sums $\sum_n \exp(-nx)$ and $\sum_n nx\exp(-nx)$. We find (see handout)

$$\sum_n \exp(-nx) = \frac{1}{1-\exp(-x)} \tag{3.7}$$

and

$$\sum_n nx\exp(-nx) = \frac{x\exp(-x)}{(1-\exp(-x))^2} \tag{3.8}$$

Therefore,

$$\langle E\rangle = kT\left(\frac{x\,\mathrm{e}^{-x}\,(1-\mathrm{e}^{-x})^{-2}}{(1-\mathrm{e}^{-x})^{-1}} + \frac{x}{2}\right) \tag{3.9}$$

$$= \frac{h\nu\exp(-x)}{1-\exp(-x)} + \frac{h\nu}{2} \tag{3.10}$$

$$= \frac{h\nu}{\mathrm{e}^{h\nu/kT}-1} + \frac{h\nu}{2} \tag{3.11}$$

We reiterate: the $h\nu/2$ term is unobservable $\Longrightarrow$ Renormalize zero-point of energy to get rid of it.

Could have "known" this result since from Bose-Einstein statistics of particles with chemical potential $\mu = 0$ the occupation number is

$$n_\gamma(\nu,T) = \frac{\langle E\rangle}{h\nu} = \frac{1}{\exp(h\nu/kT)-1} \tag{3.12}$$

To prove Eqs. (3.7) and (3.8), look at the Taylor series of $f(y) = (1-y)^{-1}$.

By induction:

$$f(y) = (1-y)^{-1} \tag{3.13}$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} = \frac{(-1)(-1)}{(1-y)^2} = \frac{1}{(1-y)^2} \tag{3.14}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} = \frac{(-1)(-2)}{(1-y)^3} = \frac{1\cdot 2}{(1-y)^3} \tag{3.15}$$

and in general

$$\frac{\mathrm{d}^n f}{\mathrm{d}y^n} = \frac{n!}{(1-y)^{n+1}} \tag{3.16}$$

Therefore, the Taylor series of $f(y)$ around $y = 0$ is

$$\frac{1}{1-y} = \sum_n \frac{1}{n!} \left.\frac{\mathrm{d}^n f}{\mathrm{d}y^n}\right|_{y=0} y^n = \sum_n y^n \tag{3.17}$$

Substituting $y = \exp(-x)$ proves Eq. (3.7).

To prove Eq. (3.8), we need to compute

$$\sum_n nx \exp(-nx) = x \sum_n n \exp(-nx) \tag{3.18}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_n \exp(-nx) = -\sum_n n \exp(-nx) \tag{3.19}$$

such that

$$\sum_n n \exp(-nx) = -\frac{\mathrm{d}}{\mathrm{d}x} \sum_n \exp(-nx) \tag{3.20}$$

by Eq. (3.7)

$$= -\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{1-\exp(-x)} \right) \tag{3.21}$$

$$= \frac{\exp(-x)}{(1-\exp(-x))^2} \tag{3.22}$$

Multiplying with x proves Eq. (3.8).

Derivation: Step 2, I

Second Step: Computation of density of phase space cells in box L_x, L_y, L_z.

Wave vector of photon:

$$\mathbf{k} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{2\pi\nu}{c}\mathbf{n} \tag{3.23}$$

To get all possible photons: count distinguishable photons at same frequency, i.e., photons with different spin or different number of nodes (=different n).

Spin is easy: there are 2 polarization states

Number of nodes: in the x, y, or z direction, number of nodes is

$$n_x = \frac{L_x}{\lambda} = \frac{k_x L_x}{2\pi} \quad \Longleftrightarrow \quad \mathrm{d}n_x = \frac{L_x}{\lambda} = \frac{L_x\,\mathrm{d}k_x}{2\pi} \tag{3.24}$$

For $n \gg 1$, can go to "continuum of states". Eq. 3.24 then implies

$$\mathrm{d}N = \mathrm{d}n_x\,\mathrm{d}n_y\,\mathrm{d}n_z = \frac{L_x L_y L_z\,\mathrm{d}^3k}{(2\pi)^3} = \frac{V\,\mathrm{d}^3k}{(2\pi)^3} \tag{3.25}$$

Therefore, the number of states per unit volume per wave number is

$$\frac{n_k}{\mathrm{d}^3k} = 2 \cdot \frac{\mathrm{d}N}{V}\frac{1}{\mathrm{d}^3k} = \frac{2}{(2\pi)^3} \tag{3.26}$$

Factor 2 from spin.

Derivation: Step 2, II

Because of Eq. (3.23),

$$\mathrm{d}^3k = k^2\,\mathrm{d}k\,\mathrm{d}\Omega = \frac{(2\pi)^3}{c^3}\nu^2\,\mathrm{d}\nu\,\mathrm{d}\Omega \tag{3.27}$$

such that the density of states

$$\rho_\mathrm{s} = \frac{n_\nu}{\mathrm{d}\nu\,\mathrm{d}\Omega} = \frac{2}{(2\pi)^3}\cdot\frac{(2\pi)^3}{c^3}\nu^2 = \frac{2\nu^2}{c^3} \tag{3.28}$$

(number of states per solid angle, per volume, per frequency).

Blackbody spectrum

To summarize, we had:

Mean energy of state:

$$\langle E\rangle = \frac{h\nu}{e^{h\nu/kT}-1} \tag{3.11}$$

State density:

$$\rho_\mathrm{s} = \frac{2\nu^2}{c^3} \tag{3.28}$$

The total energy density is then

$$u_\nu(\Omega) = \langle E\rangle \cdot \rho_\mathrm{s} \tag{3.29}$$

$$= \frac{2h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT)-1} \tag{3.30}$$

(energy per volume per frequency per solid angle)

Because of Eq. (2.30) ($u_\nu = I_\nu/c$), the intensity is given by

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1} =: B_\nu \tag{3.31}$$

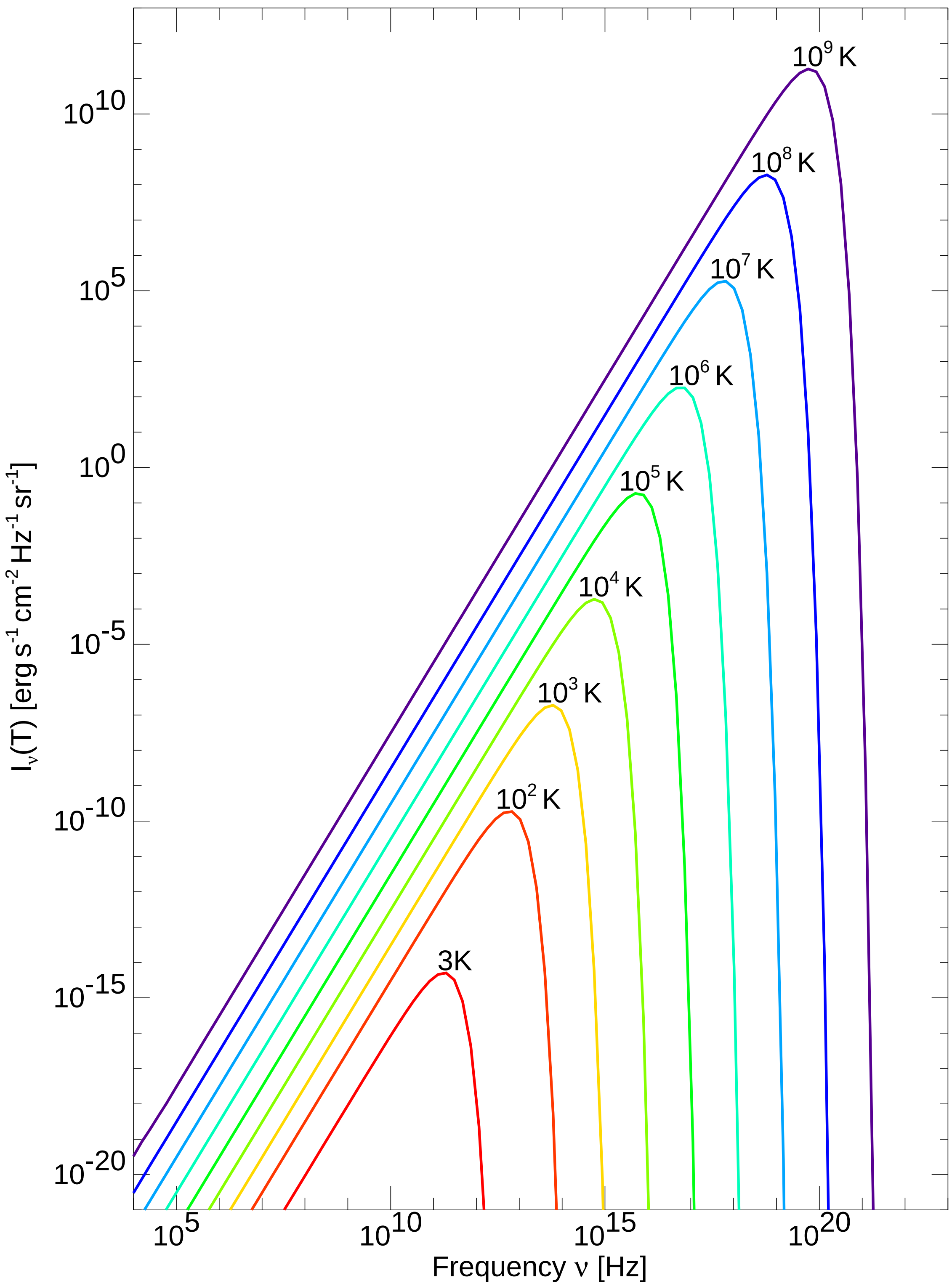
This is the spectrum of a black body.

In λ space, the spectrum is

$$B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT)-1} \tag{3.32}$$

(since we need $B_\lambda\,\mathrm{d}\lambda = B_\nu\,\mathrm{d}\nu$).

Spectrum


10^9 K
10^8 K
10^7 K
10^6 K
10^5 K
10^4 K
10^3 K
10^2 K
3K
10^{10}
10^5
10^0
10^{-5}
10^{-10}
10^{-15}
10^{-20}
$I_\nu(T)$ [erg s^{-1} cm^{-2} Hz^{-1} sr^{-1}]
10^5
10^{10}
10^{15}
10^{20}
Frequency ν [Hz]

Rayleigh-Jeans Law

For $h\nu \ll kT$ ($\nu \lesssim 2 \times 10^{10} T$),

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + \dots \tag{3.33}$$

such that

$$B_\nu \approx \frac{2\nu^2}{c^2} kT \tag{3.34}$$

This is the Rayleigh-Jeans law.

The Rayleigh-Jeans law is used in the radio regime to define the brightness temperature,

$$T_\mathrm{b} = I_\nu \cdot \frac{c^2}{2k\nu^2} \tag{3.35}$$

where I_ν is the measured radio intensity.

Wien Spectrum

For $h\nu \gg kT$, ($\nu \gtrsim 2 \times 10^{10} T$),

$$\exp\left(\frac{h\nu}{kT}\right) - 1 \sim \exp\left(\frac{h\nu}{kT}\right) \tag{3.36}$$

such that

$$B_\nu \approx \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{kT}\right) \tag{3.37}$$

the Wien spectrum (or Wien's law).

Wien Displacement Law

The frequency of maximum intensity, ν_{max} is obtained by solving

$$\left.\frac{\partial B_\nu}{\partial \nu}\right|_{\nu=\nu_{\mathrm{max}}} = 0 \qquad (3.38)$$

which is equivalent to solving

$$x = 3(1 - \exp(-x)) \qquad (3.39)$$

where $x = h\nu_{\mathrm{max}}/kT$. Numerically, $x = 2.82$, therefore

$$h\nu_{\mathrm{max}} = 2.82 \cdot kT \qquad (3.40)$$

This is the Wien displacement law.

The frequency of maximum flux is directly proportional to the black body temperature.

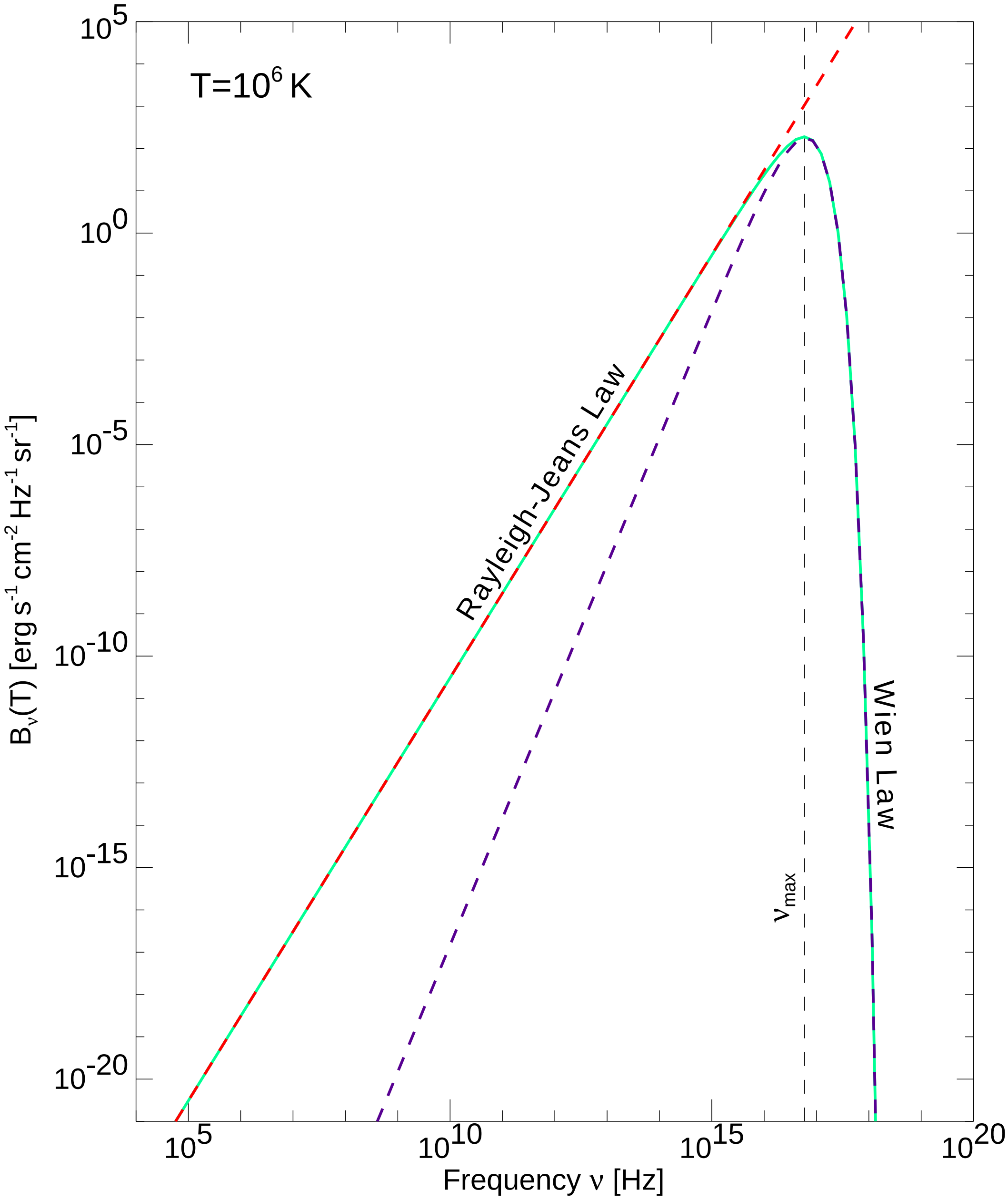
Likewise, for B_λ, one finds

$$\lambda_{\mathrm{max}} T = 0.2898\,\mathrm{cm\,K} \qquad (3.41)$$

Note that $\lambda_{\mathrm{max}}\nu_{\mathrm{max}} \neq c$!

Do not confuse Wien's law and the Wien displacement law...

Summary: Rayleigh-Jeans vs. Wien



Rayleigh-Jeans applies for $\nu \lesssim \nu_{\rm max}$

Wien applies for $\nu \gtrsim \nu_{\rm max}$.

Stefan-Boltzmann law

The total brightness of a black body is obtained from

$$B(T) = \int_0^\infty B_\nu(T)\,\mathrm{d}\nu \tag{3.42}$$

. . . substituting $x = h\nu/kT$

$$= \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3\,\mathrm{d}x}{\exp(x)-1} \tag{3.43}$$

. . . the integral has the value $\pi^4/15$

$$= \frac{2\pi^4 k^4}{15c^2h^3}T^4 = \frac{ac}{4\pi}T^4 = \frac{\sigma_{\mathrm{SB}}T^4}{\pi} \tag{3.44}$$

Convert the brightness to the flux ($F = \pi B$, Eq. 2.24), to obtain

$$F = \sigma_{\mathrm{SB}}T^4 \tag{3.45}$$

the Stefan-Boltzmann law.

And, yes, Boltzmann's fi rst name is Ludwig, while Stefan's fi rst name is Josef.

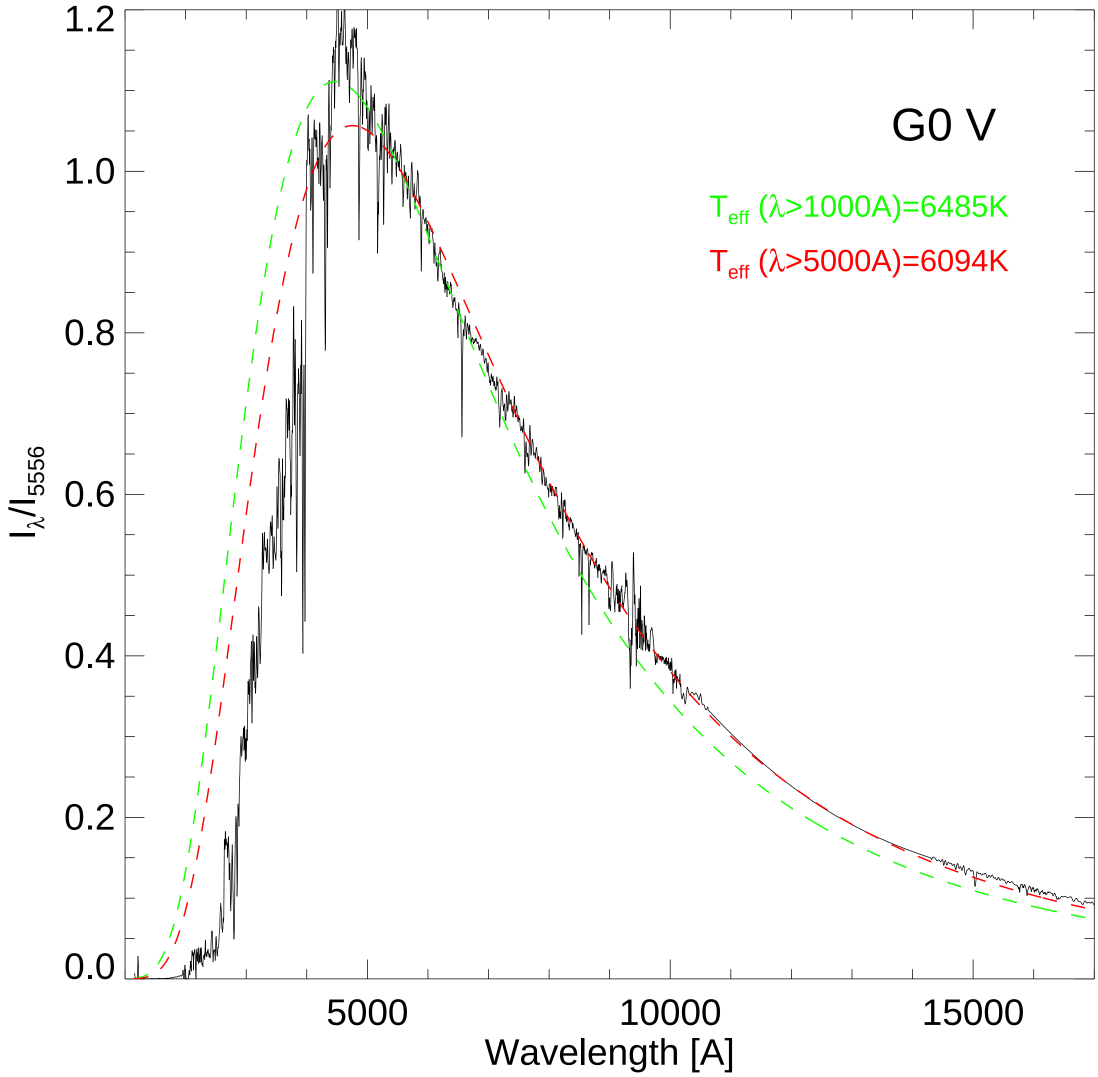
a is the radiation density constant,

$$a := \frac{8\pi^5 k^4}{15c^3h^3} = 7.566 \times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}} \tag{3.46}$$

also written as the Stefan-Boltzmann constant

$$\sigma_{\mathrm{SB}} := \frac{2\pi^5 k^4}{15c^2h^3} = 5.671 \times 10^{-5}\,\mathrm{erg\,cm^{-2}\,K^{-4}\,s} \tag{3.47}$$

Effective Temperature



G0 V spectrum after Pickles (1998), PASP 110, 863

The effective temperature, T_{eff}, of a spectrum I_ν is the temperature where

$$F = \int I_\nu \cos\theta \, d\nu \, d\Omega = \sigma T_{\text{eff}}^4 \tag{3.48}$$

Sometimes, I_ν is only known over a certain wavelength range, and depending on the spectrum the *measured* T_{eff} will depend on this range (see fi gure).

Application: Planets

The temperature of an irradiated body is given from energy equilibrium:

$$\frac{L_\odot}{4\pi a^2}\pi r^2 = \sigma_{\text{SB}} T^4 4\pi r^2 \qquad (3.49)$$

where a: distance to sun, r: planetary radius.

Therefore

$$T = \left(\frac{L_\odot}{16\pi\sigma_{\text{SB}} r^2}\right)^{1/4} = \frac{281\,\text{K}}{(a/1\,\text{AU})^{1/2}} \qquad (3.50)$$

Last step used $L_\odot = 4 \times 10^{33}\,\text{erg}\,\text{s}^{-1}$ and $1\,\text{AU} = 1.496 \times 10^{13}\,\text{cm}$.

If the planet reflects part of the radiation and if the IR emissivity is only roughly a BB, then Eq. (3.49) is modified,

$$(1-B)\frac{L_\odot}{4\pi a^2}\pi r^2 = \epsilon\sigma_{\text{SB}} T^4 4\pi r^2$$

$$\Longrightarrow T = \frac{281\,\text{K}}{(a/1\,\text{AU})^{1/2}}\left(\frac{1-B}{\epsilon}\right)^{1/4} \qquad (3.51)$$

where B: Bond albedo, and ϵ: effective emissivity

For the Earth, $B = 0.39$, for Venus, $B = 0.72$. Thus, since $T_{\text{Earth}} \sim 288\,\text{K}$, $\epsilon_{\text{Earth}} = 0.55 < 1$ (greenhouse effect).

If the planet is not a fast rotator, replace $4\pi r^2$ by $2\pi r^2$.