Quantization of angular momentum $\vec{L}$ magnitude: 8 Mar 2007 - EJZ

$$E = T + V = \frac{p_r^2}{2\mu} + \frac{L^2}{2\mu r^2} + V(r)$$

Tipler-Llewellyn (7-17) 298

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}, \quad \hat{p}_{x_i}^2 = (-i\hbar)^2\frac{\partial^2}{\partial x_i^2} = -\hbar^2\nabla_{x_i}^2$$

(momentum operators)

□ Spherical Coordinates: $\hat{p}_r = (-\hbar)^2\nabla_r^2 =$ __

$$\hat{p}_z = -i\hbar\frac{\partial}{\partial z} \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}, \quad \hat{L}^2 = -\hbar^2(\nabla_\varphi^2 + \nabla_\theta^2) = \_$$

□

(7-20)

Find $\hat{L}^2$ in Schrödinger Eqn in Spherical Coordinates: (7-9) 294

□
$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{\hbar^2}{2\mu r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] + V(r)\psi = E\psi$$

Separate variables: Let $\psi(r,\theta,\varphi) = R(r)f(\theta)g(\varphi)$. Then Sch. Eqn becomes

(7-12) 295

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu r^2}{\hbar^2}(E - V(r)) =$$
$$-\left[\frac{1}{f\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) + \frac{1}{g\sin^2\theta}\frac{d^2g}{d\varphi^2}\right] = \text{constant} = l(l+1)$$

□ Multiply $[\sim] = l(l+1)$ by $\hbar^2 f(\theta)g(\varphi) = \hbar^2 Y_{lm}(\theta,\varphi)$ (Spherical harmonics) 296-297

$l = 0,1,2,3\cdots$, $m = 0,\pm1,\cdots,\pm l$

(7-21a) 298

Same as $\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1)Y_{lm}(\theta,\varphi)$

or, since $\psi(r,\theta,\varphi) = R(r)Y(\theta,\varphi)$, $\hat{L}^2\psi(r,\theta,\varphi) = \hbar^2 l(l+1)\psi(r,\theta,\varphi)$

□ Therefore, eigenvalues are $L =$ __

Space-quantization of angular momentum L_z:

- $\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi} \rightarrow \hat{L}_z^2 =$

- Find $\hat{L}_z^2$ in Schrödinger Eqn in spherical coords (7-9) 294

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)-\frac{\hbar^2}{2\mu r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+V(r)\psi=E\psi$$

(shown last time)
Angular part of the equation separates to (7.13) 296

$$\underbrace{-m^2 = \frac{1}{g(\phi)}\frac{d^2 g(\phi)}{d\phi^2}}\;=\;\underbrace{-l(l+1)\sin^2\theta-\frac{\sin\theta}{f(\theta)}\frac{d}{d\theta}\left[\sin\theta\frac{df(\theta)}{d\theta}\right]}_{RHS}$$

- Solve for $g(\phi)$:

"The single-valued condition on ψ implies that $g(\phi+2\pi)=g(\phi)$, which requires that m be an integer ($\pm$)" p.296

It can be shown that the solution to [RHS] $=-m^2$ is (7-15) 296)

$f(\theta)$ = Legendre functions

$$f(\theta)=\frac{(\sin\theta)^{|m|}}{2^l l!}\left[\frac{d}{d\cos\theta}\right]^{l+|m|}(\cos^2\theta-1)^l$$

Limits on l and m: find $f(\theta=0)$ and $f(\pi)$. What is required to keep $f(\theta)$ FINITE?

$Y_{lm}(\theta,\phi)=f(\theta)\,g(\theta)$ = spherical harmonics = soln for ANY SPHERICAL $V(r)$