Quantization of angular momentum L magnitude: 8 Mar 2007 - EJZ

$$E = T + V = \frac{p_r^2}{2\mu} + \frac{L^2}{2\mu r^2} + V(r)$$ Tipler-Llewellyn (7-17) 298

$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$, $\hat{p}_{x_i}^2 = (-i\hbar)^2 \frac{\partial^2}{\partial x_i^2} = -\hbar^2 \nabla_{x_i}^2$ (momentum operators)

Spherical coordinates: $\hat{p}_r = (-\hbar)^2 \nabla_r^2 = -\hbar^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$

$\hat{p}_z = -i\hbar \frac{\partial}{\partial z}$ $\quad z\ \phi \quad$ $\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$, $\hat{L}^2 = -\hbar^2(\nabla_\phi^2 + \nabla_\theta^2) =$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

(7-20)

Find $\hat{L}^2$ in Schrödinger Eqn in Spherical Coordinates: (7-9) 294

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \underbrace{- \frac{\hbar^2}{2\mu r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]}_{\frac{L^2\psi}{2\mu r^2}} + V(r)\psi = E\psi$$

Separate variables: Let $\psi(r,\theta,\phi) = R(r) f(\theta) g(\phi)$. Then Sch. Eqn becomes

(7-12) 295

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu r^2}{\hbar^2}(E - V(r)) =$$
$$-\left[\frac{1}{f\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) + \frac{1}{g\sin^2\theta}\frac{d^2g}{d\phi^2}\right] = \text{constant} = l(l+1)$$

296-297

Multiply $[\sim] = l(l+1)$ by $\hbar^2 f(\theta) g(\phi) = \hbar^2 Y_{lm}(\theta,\phi)$ (Spherical harmonics $l = 0,1,2,3\ldots$, $m = 0, \pm1, \ldots, \pm l$)

(7-21a) 298

$$-\hbar^2\left[\frac{fg}{f\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) + \frac{fg}{g\sin^2\theta}\frac{d^2g}{d\phi^2}\right] = \hbar^2 l(l+1) f(\theta) g(\phi)$$

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right)g + \frac{1}{\sin^2\theta}\frac{d^2g}{d\phi^2}f\right] = \hbar^2 l(l+1) Y_{lm}(\theta,\phi)$$

Same as $\hat{L}^2 Y_{lm}(\theta,\phi) = \hbar^2 l(l+1) Y_{lm}(\theta,\phi)$

or, since $\psi(r,\theta,\phi) = R(r) Y(\theta,\phi)$, $\hat{L}^2\psi(r,\theta,\phi) = \hbar^2 l(l+1)\psi(r,\theta,\phi)$

Therefore, eigenvalues are $L = \sqrt{\hbar^2 l(l+1)} = \hbar\sqrt{l(l+1)}$

Space-quantization of angular momentum L_z:

- $\hat{L}_z = -i\hbar \frac{\partial}{\partial\phi} \rightarrow \hat{L}_z^2 = -\hbar^2 \frac{\partial^2}{\partial\phi^2}$

- Find $\hat{L}_z^2$ in Schrödinger Eqn in spherical coords (7-9) 294

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{\hbar^2}{2\mu r^2}\underbrace{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]}_{\frac{L_z^2\psi}{2\mu r^2}?} + V(r)\psi = E\psi$$

(skipping...)
Angular part of the equation separates to (7.13) 296

$$\underbrace{-m^2 = \frac{1}{g(\phi)}\frac{d^2 g(\phi)}{d\phi^2}}\ = \underbrace{-l(l+1)\sin^2\theta - \frac{\sin\theta}{f(\theta)}\frac{d}{d\theta}\left[\sin\theta\frac{df(\theta)}{d\theta}\right]}_{RHS}$$

- Solve for $g(\phi)$: $\frac{d^2g}{d\phi^2} = -mg^2$; what functions $\simeq$ their own second deriv?
  sin, cos, exp

Try $\underline{g = e^{im\phi}}$, then $\frac{dg}{d\phi} = ime^{im\phi}$

$$\frac{d^2g}{d\phi^2} = (im)^2 e^{im\phi} = -m^2 e^{im\phi} = -m^2 g \checkmark$$

"The single-valued condition on ψ implies that $g(\phi+2\pi) = g(\phi)$, which requires that m be an integer ($\pm$)" p.296

It can be shown that the solution to [RHS] $= -m^2$ is
$f(\theta)$ = Legendre functions (7-15) 296)

$$f(\theta) = \frac{(\sin\theta)^{|m|}}{2^l l!}\left[\frac{d}{d\cos\theta}\right]^{l+|m|}(\cos^2\theta - 1)^l$$

Limit on l and m: find $f(\theta=0)$ and $f(\pi)$. What is required to keep $f(\theta)$ FINITE?

$l = 0, 1, 2, 3, \ldots \quad m = 0, \pm 1, \ldots, \pm l$

$Y_{lm}(\theta, \phi) = f(\theta)\, g(\theta)$ = spherical harmonics = solns for ANY SPHERICAL $V(r)$