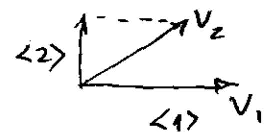
MATRIX IDENTITIES

$$[\Omega, \Lambda] = -[\Lambda, \Omega]$$
$$[\Omega, \Lambda\theta] = \Lambda[\Omega, \theta] + [\Omega, \Lambda]\theta$$
$$[\Omega\Lambda, \theta] = \Omega[\Lambda, \theta] + [\Omega, \theta]\Lambda$$
$$[\Omega, \Lambda+\theta] = [\Omega, \Lambda] + [\Omega, \theta]$$
$$[\Omega+\Lambda, \theta] = [\Omega, \theta] + [\Lambda, \theta]$$

GRAM-SCHMIDT TH: constructing orthonormal $|1\rangle$, $|2\rangle$, $|3\rangle$ from vectors $|V_1\rangle$, $|V_2\rangle$, $|V_3\rangle$

$$|1\rangle = |V_1\rangle$$
$$|2\rangle = |V_2\rangle - \frac{|1\rangle\langle 1|V_2\rangle}{\langle 1|1\rangle}$$
$$|3\rangle = |V_3\rangle - \frac{|1\rangle\langle 1|V_3\rangle}{\langle 1|1\rangle} - \frac{|2\rangle\langle 2|V_3\rangle}{\langle 2|2\rangle}$$

etc.



MATRIX PROPERTIES

Hermitian $\Omega = \Omega^\dagger$

Commute $[\Omega, \theta] = \Omega\theta - \theta\Omega = 0 = -[\Omega, \theta] = [\theta, \Omega]$

Unitary $UU^\dagger = I$

FINDING E-VALUES & E-VECTORS of MATRIX Ω:

e-values ω satisfy $0 = \det(\Omega - \omega I)$

e-vectors $|\omega = i\rangle$ satisfy $(\Omega - \omega I)|\omega = i\rangle = 0$

REVIEW - EARLY CHAPTERS

inner product satisfies $\langle V_i | V_i \rangle \geq 0$

$$\langle V_i | V_j \rangle = \langle V_j | V_i \rangle^*$$

linear (2nd) $\langle V_i | \alpha V_j + \beta V_k \rangle = \alpha \langle V_i | V_j \rangle + \beta \langle V_i | V_k \rangle$

antilinear (1st) $\langle \alpha V_i + \beta V_j | V_k \rangle = \alpha^* \langle V_i | V_k \rangle + \beta^* \langle V_j | V_k \rangle$

orthonormal : $\langle e_i | e_j \rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$\langle V | V' \rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} v_i^* v_j' \delta_{ij} = \sum_{i=1}^{3} v_i^* v_j' = \int v_i^* v_j \, dx \quad (V(x))$$

$\langle j | i' \rangle = \langle j | \Omega | i \rangle = \Omega_{ji}$ $\quad |i'\rangle = \Omega |i\rangle$ $\quad \Omega_{ji} \leftrightarrow \begin{array}{cc} & i \\ j & \left(\Omega \right) \end{array}$

$\Omega^\dagger_{ji} = \Omega^*_{ij}$ dagger = transpose conjugate

unitary $UU^\dagger = I \rightarrow U^\dagger U = I$

basis of operator is its eigenfxns, or it eigenvalues diagonalized by the matrix of eigenvalues.

commuting Hermitian operators have common diagonalizing basis

p.50
$$\ddot{x}_1 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2 \qquad \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \Omega_{11} = \Omega_{22} = \frac{-2k}{m}$$

$$\ddot{x}_2 = \frac{k}{m} x_1 - \frac{2k}{m} x_2 \qquad \Omega_{12} = \Omega_{21} = \frac{k}{m}$$

$$|\ddot{x}(t)\rangle = \Omega |x(t)\rangle$$

modes $|1\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ m_1 displaced, m_2 still $\qquad |2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftrightarrow$ m_1 still, m_2 displaced

arbitrary state : $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$

$$|x\rangle = |1\rangle x_1 + |2\rangle x_2$$

In $|1\rangle, |2\rangle$ basis, $\Omega \leftrightarrow \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix}$ but components x_1 & x_2 obey coupled eqns. So Diagonalize → uncouple

ADDITION of vectors → vector
INNER PRODUCT of vectors → scalar. Must obey LINEARITY.
SCALAR MULTIPLICATION of vector → vector

all vectors in a vector space $\mathbb{V}^n$ can be written as a linear combination of some of the n basis vectors $\vec{V}_i$ (LI set) : $\vec{V} = \Sigma \alpha_i \vec{V}_i$
components ↗ = expand by LINEAR EXPANSION

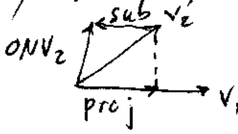
Orthonormal vectors: $\langle e_i | e_j \rangle = \delta_{ij}$
General: $\langle V_0 | V_0' \rangle = \Sigma_i V_i V_i'$ sum of products of components

GRAM SCHMIDT: make $V_1 = \frac{V_1'}{|V_1'|} = |1\rangle$

$$V_2 = V_2' - \frac{|V_1'\rangle\langle V_1' | V_2'\rangle}{\langle V_1' | V_1'\rangle}$$

$$V_3 = V_3' - \frac{|V_1'\rangle\langle V_1' | V_3'\rangle}{\langle V_1' | V_1'\rangle} - \frac{|V_2'\rangle\langle V_2' | V_3'\rangle}{\langle V_2' | V_2'\rangle} \quad \text{etc.}$$

where we started with non-orthonormal V_1', V_2', V_3' and get ON by subtracting from each original its projection on previous ON's.



COMPLEX VECTOR SPACE: components, scalars are complex

NOTE: INNER PRODUCT IS NO LONGER COMMUTATIVE: $\langle V | V' \rangle = \langle V' | V \rangle^*$
complex conjugate: where $c = a + ib$, $c^* = a - ib$.
but NORM IS STILL A POSITIVE SCALAR

→ LINEAR dot product: $\langle V | \alpha V' + \beta V'' \rangle = \alpha \langle V | V' \rangle + \beta \langle V | V'' \rangle$

→ ANTILINEAR when constant in first term: $\langle \alpha V | V' \rangle = \alpha^* \langle V | V' \rangle$
$\langle \alpha V' + \beta V'' | V \rangle = \alpha^* \langle V' | V \rangle + \beta^* \langle V'' | V \rangle$

→ INNER PRODUCT: $\langle V | V' \rangle = \Sigma_i V_i^* V_i' = \underbrace{[V_1^*, V_2^*, \ldots]}_{\langle V| \text{ bra}} \underbrace{\begin{bmatrix} V_1' \\ V_2' \\ \vdots \end{bmatrix}}_{|V'\rangle \text{ ket}}$

TAKING ADJOINT OF VECTOR V: V^{T*} (T: turn it on side)

$$|v'''\rangle = \alpha|v\rangle + \beta|v\rangle\langle v'|v''\rangle + \cdots$$
$$\langle v'''| = \alpha^*\langle v| + \beta^*\langle v''|v'\rangle\langle v| + \cdots$$

component of vector $v_j = \langle j|v\rangle$ projection on basis vector j.

sum of projections · ~~components~~ directions $|v\rangle = \sum_i |i\rangle\langle i|v\rangle$

MATRIX REPRESENTATION: $[v] = \begin{bmatrix}1\\0\\ \vdots\end{bmatrix}\langle 1|v\rangle + \begin{bmatrix}0\\1\\ \vdots\end{bmatrix}\langle 2|v\rangle + \cdots$

$$= \begin{bmatrix}\langle 1|v\rangle\\ \langle 2|v\rangle\\ \vdots\end{bmatrix}$$

$|v\rangle = \sum_i |i\rangle\langle i|v\rangle$ and $\langle v'| = \sum_j \langle v'|j\rangle\langle j|$

$$\to \langle v'|v\rangle = \left(\sum_j \langle v'|j\rangle\langle j|\right)\left(\sum_i |i\rangle\langle i|v\rangle\right)$$
$$= \sum_j\sum_i \langle v'|j\rangle \underbrace{\langle j|i\rangle}_{\delta_{ji} = 1 \text{ when } j=i} \langle i|v\rangle$$
$$= \sum_i \langle v'|j\rangle \langle i|v\rangle$$

OPERATORS: LINEAR: can bring things inside summation sign

$$\Omega|\alpha V\rangle = \alpha\Omega|V\rangle$$
$$\Omega(|V_1\rangle + |V_2\rangle) = \Omega|V_1\rangle + \Omega|V_2\rangle$$

$$|v'\rangle = \Omega|v\rangle = \Omega\left(\sum_i |i\rangle\langle i|v\rangle\right) = \sum_i \underbrace{\Omega|i\rangle}_{|i'\rangle \text{ transformed basis vector}} \langle i|v\rangle$$

$\langle j|v'\rangle = \sum_i \langle j|\Omega|i\rangle\langle i|v\rangle$ j^{th} component of V'

$$\underbrace{\begin{bmatrix} v_1' \\ v_2' \\ \vdots \end{bmatrix}}_{\text{all } v_j'\text{'s}} = \begin{bmatrix} \langle 1|\Omega|1\rangle & \langle 1|\Omega|2\rangle & \\ \langle 2|\Omega|1\rangle & \ddots & \\ & \langle j|\Omega|i\rangle & \end{bmatrix} \underbrace{\begin{bmatrix} \langle 1|v\rangle \\ \langle 2|v\rangle \\ \vdots \end{bmatrix}}_{V}$$

$$\underbrace{\left(\sum_i |i\rangle\langle i|\right)}_{\text{Identity operator}} |v\rangle = |v\rangle$$

Operators - transform one vector into another $\Omega|V\rangle \to |V'\rangle$

Multiplication of operators - transform a vector by one operator after another $\Lambda\Omega|V\rangle \to |V''\rangle$

Components of matrix which is product of operators: $\langle i|\Lambda\Omega|j\rangle = \sum_k \langle i|\Lambda|k\rangle\langle k|\Omega|j\rangle$

Multiplication of operators doesn't commute: $[\Omega,\Lambda] = \Omega\Lambda - \Lambda\Omega$
derivative-like: $[\Omega, \Lambda\Theta] = \Lambda[\Omega,\Theta] + [\Omega,\Lambda]\Theta$

adj UNITARY TRANSFORMATION $\underline{U|i\rangle = |i'\rangle}$ where $i'j' = \delta_{ij}$
transforms orthonormal basis vectors into another basis set
eg all rotations are unitary; some reflections

Ex transformation matrix operates on x:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

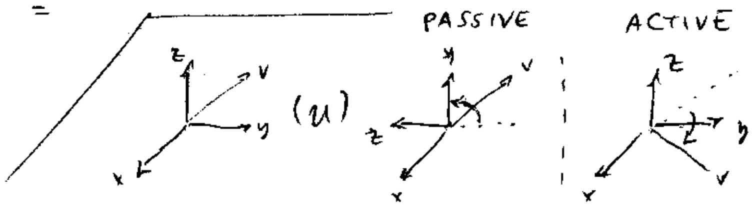
transforms x to y

but NOT $\langle i|U \to \langle i'|$, so

adj To get bras: $\underline{\langle v|\Omega^\dagger = \langle v'|}$: $\langle i|\Omega^\dagger|j\rangle = \langle i'|j\rangle = \langle j|i'\rangle^* = \langle j|\Omega|i\rangle^*$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \checkmark$$

$(\Lambda\Omega)^\dagger = \Omega^\dagger\Lambda^\dagger$



To find $\underline{\text{components } \langle i'|v\rangle}$ of v in a new basis after U trans:

v in new basis $\langle i'|v\rangle = \langle i|U^\dagger|v\rangle = \sum_j \langle i|U^\dagger|j\rangle\langle j|v\rangle$ ($U^\dagger$ in old basis, v in old basis)

PASSIVE: Change basis, leave vector; ACTIVE: Change vector, leave basis.

TRANSFORMING TO NEW BASIS

components of matrix transformed to new basis :

$$\langle i'|\Omega|j'\rangle = \langle i|U^{\dagger}\Omega U|j\rangle$$

$$= \sum_k \sum_m \langle i|U^{\dagger}|k\rangle\langle k|\Omega|m\rangle\langle m|U|j\rangle$$

multiply elements of these three operators

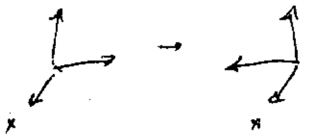
$$\Omega_{new} = U^{\dagger}\Omega_{old} U$$

Sat 8 Jan

EIGENVALUE - EIGENVECTOR

If the operation of Ω on v recovers scalar ω multiple of v then

$$\underbrace{\Omega|v\rangle = \omega}_{\text{eigenvalue}}\,|v\rangle$$ eigenvector : direction only

eg rotation  eigenvector x , eigenvalue 1
(evec 2x , eval 2 possible too)

$$(\Omega - \omega I)|v\rangle = |0\rangle \Rightarrow \det(\Omega - \omega I) = 0$$

solve for n ω's, ~~use to~~ get an evec. for each eval. ω

to find inverse of 

get $\dfrac{\det m}{\det \text{whole}}$

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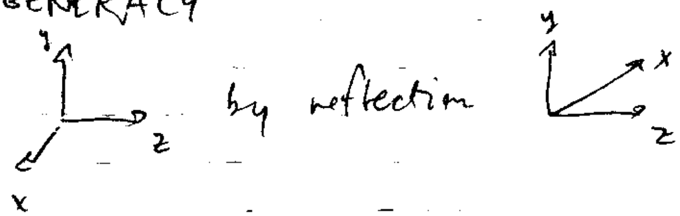
DEGENERACY

transform  by reflection

and everything in y-z plane is degenerate.

Degenerate operator doesn't specify eigenvalues in degenerate subspace.

So here we pick any two orthogonal vectors in y-z space

Tu. 11. Jan. 83

Hermitian operator $\Omega^\dagger = \Omega$ corresponds to real number $\alpha^* = \alpha$.
(real eigenvalues)
$U^\dagger U = I$ corresponds to $\alpha^* \alpha = |\alpha|^2 = 1$. Only switch bases.

$\Omega^\dagger = -\Omega$ antihermitian: imaginary $\alpha^* = -\alpha$
will also have orthogonal bases

Recall: every observable associated with operator,
possible values correspond with eigenvalues of operator
Operator must be hermitian, for measured values must be real.

DIAGONALIZATION

Rewrite Ω in its own eigenbasis → it will become diagonal with eigenvalues

repeating: can convert old basis ⟷ new basis

$$\langle i'|\Omega|j'\rangle = \langle i|U^\dagger \Omega U|j\rangle = \sum_k \sum_m \langle i|U^\dagger|k\rangle\langle k|\Omega|m\rangle\langle m|U^\dagger|j\rangle$$

$U^\dagger \Omega U \rightarrow$ diagonal

$$\underset{\text{old}}{\Omega} \rightarrow \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad |\omega = 1\rangle \leftrightarrow \begin{bmatrix} i \\ 1 \end{bmatrix}\sqrt{\tfrac{1}{2}}, \quad |\omega = 1\rangle \leftrightarrow \sqrt{\tfrac{1}{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

in new basis: $\begin{bmatrix} \langle \omega = 1|\omega = 1\rangle \\ \langle \omega = -1|\omega = 1\rangle \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

NEW BASIS $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ +1 \end{bmatrix}$ for every old Ω

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vice versa: $\underset{\text{new}}{\langle j'|}\underset{\text{old}}{i\rangle} \rightarrow \langle j|U^\dagger|i\rangle$

~~New~~ basis vectors in terms of old: $\sqrt{\tfrac{1}{2}}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ i \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -i \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Uncoupled DIFFEQ for eq. MOTION corresponds to Diagonalization of matrix (waves example)

Diagonalization can be done by just putting evals on diagonal and set evecs as $\longleftrightarrow$ I.

~~wunta~~ sim. diagonalizable?

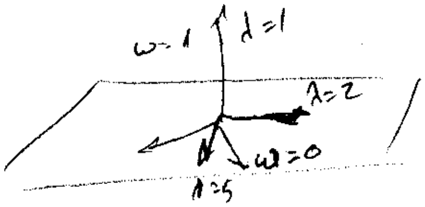
$\Omega \leftrightarrow \sqrt{\tfrac{1}{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad \Lambda \leftrightarrow \sqrt{\tfrac{1}{2}}\begin{pmatrix}3&0&0\\0&3&0\\0&0&3\end{pmatrix}$

commutator $|\Omega, \Lambda| = \Omega\Lambda - \Lambda\Omega = \begin{bmatrix}0&0&0\\1&0&1\\0&0&0\end{bmatrix} - ($

oh well, they commute since ~~ee~~ ebas of Λ is I and that is ebas of all operators
And any basis is an ebas for I.

So just choose a matrix that makes Ω diagonal.

Use Λ to pick ebas - can't wit Ω:



Want to find unique ebas.

Try Shankar's 3×3 example.

find ebas for undegenerate →