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$\boxed{\text{QM Ch 13: H atom}}$ #2 p. 363 (1)

So far we have, for $V(r)$, $\psi = R_{E\ell}(r)\, Y_\ell^m(\theta,\varphi)$

where $R_{E\ell}(r) = \dfrac{U_{E\ell}(r)}{r}$ and $U_{E\ell} \xrightarrow[r\to 0]{} r^{\ell+1}$, $U_{E\ell} \xrightarrow[r\to\infty]{} e^{-\rho}$, $\rho = \sqrt{2mW}\,\dfrac{r}{\hbar}$

① $\left\{\dfrac{d^2}{dr^2} + \dfrac{2m}{\hbar^2}\left[E + \dfrac{e^2}{r} - \dfrac{\ell(\ell+1)\hbar^2}{2mr^2}\right]\right\} U_{E\ell} = 0$ (13.1.2)

Coulomb $V(r)$

(13.1.7)

Now let $r R(r) = U_{E\ell}(r) \underset{②}{=} e^{-\rho(r)} V_{E\ell}(r)$ and try to find V.

Sub ② into ① and get (13.1.8) $\dfrac{d\rho}{dr} =$ ______

(a) $\dfrac{dU}{dr} = \dfrac{dU}{d\rho}\dfrac{d\rho}{dr} =$

$\dfrac{d^2U}{dr^2} = \dfrac{d}{dr}\left(\dfrac{dU}{dr}\right) = \dfrac{d\rho}{dr}\dfrac{d}{d\rho}\Big($

(13.1.8) $\dfrac{d^2V}{d\rho^2} - 2\dfrac{dV}{d\rho} + \left[\dfrac{e^2\lambda}{\rho} - \dfrac{\ell(\ell+1)}{\rho^2}\right] = 0$ where $\lambda = \sqrt{\dfrac{2m}{\hbar^2 W}}$

Try a series solution for $V = \rho^{\ell+1}\displaystyle\sum_{k=0}^{\infty} C_k \rho^k$

and get recursion relation $\dfrac{C_{k+1}}{C_k}$ (13.1.11)

#13.1.1 Derive 13.1.11 from 13.1.8, and 13.1.14 from
367 $\frac{C_{k+1}}{C_k}$ $\frac{d^2v}{d\rho^2}$ ... 13.1.10

Plug $v = \sum_{k=0}^{\infty} C_k \rho^{l+1+k}$ into $\frac{d^2v}{d\rho^2} - 2\frac{dv}{d\rho} + \left[\frac{e^2\lambda}{\rho} - \frac{l(l+1)}{\rho^2}\right]v = 0$

$\frac{dv}{d\rho} =$

$\frac{d^2v}{d\rho^2} =$

$v\left[\frac{e^2\lambda}{\rho} - \frac{l(l+1)}{\rho^2}\right] =$

(13.1.8) becomes:

3

Now define $m = k-1$, $m+1 = k$, rewrite 3:

$$0 = \sum_{\substack{m=-1 \\ (k=0)}}^{\infty} C_{m+1}\{(l+m+2)(l+m+1) - l(l+1)\}\rho^{l+m} + \sum_{k=0}^{\infty} C_k\{e^2\lambda - 2(l+k+1)\}\rho^{l+k}$$

We can rename this m to k and gather terms of ρ^{l+k} together:

$k=0$ or $m=-1$ term

$$0 = C_0\{(l+1)l - l(l+1)\}\rho^{l-1} + \sum_{k=0}^{\infty} \rho^{l+k}\left[C_{k+1}\{(l+k+2)(l+k+1) - l(l+1)\}\right.$$
$$+ C_k\{e^2\lambda$$

$\frac{C_k+1}{C_k} =$

Energy Levels of H-atom:

We found that $\frac{C_{k+1}}{C_k} = \frac{-e^2\lambda + 2(k+l+1)}{(k+l+2)(k+l+1) - l(l+1)} \xrightarrow[k\to\infty]{} \frac{2}{k}$

describes $v_{El} = \rho^{l+1} \sum_{k=0}^{\infty} C_k \rho^k \rightarrow \rho^m e^{2\rho}$

$u \underset{\rho\to\infty}{\sim} v e^{-\rho} \sim \rho^m e^{-\rho} e^{2\rho} \sim \rho^m e^{\rho}$ must terminate at some k.

Need $-e^2\lambda + 2(k+l+1) = 0$ at some k

(13.1.9) $\lambda^2 = \frac{2m}{\hbar^2 W}$: SOLVE for $W = E(m, \hbar, e, k, l)$

(13.1.14) $W =$

Write this in terms of the principal quantum number
$n = k + l + 1$

$E_n =$

At each n, the allowed values of l are (13.1.7):

* 13.1.3. Starting from the recursion relation, find ψ_{210}.

$$\psi_{E\ell m}(\vec{r}) = R_{E\ell}(r)\, Y_\ell^m(\theta,\varphi) = \frac{u_{E\ell}(r)}{r}\, Y_\ell^m(\theta,\varphi)$$

$u_{E\ell}(\rho) = V_{n\ell}(\rho)\, e^{-\rho}$ where $\rho = \frac{r}{n a_0}$ from (13.1.23)+(13.1.24)

$$V_{n\ell}(\rho) = \sum_{k=0}^{\infty} C_k\, \rho^{\ell+k+1}$$

$$\frac{C_{k+1}}{C_k} = \frac{[-2n] + 2(k+\ell+1)}{(k+\ell+2)(k+\ell+1) - \ell(\ell+1)}$$

[From (13.1.11) with $\lambda = \sqrt{\frac{2m}{\hbar^2 W}}$ = =

(13.1.9) (13.1.16)

So $e^2\lambda =$ $= 2n$]

ψ_{210} : $n =$ __ $\ell =$ __ $m =$ __

$\frac{C_1}{C_0} =$

Therefore all $C_k = 0$ except C_0. Just leave C_0 as an unknown constant for now.

$V_{21}(\rho) =$

$u_{21}(r) =$

Normalizing Ψ_{210}: The normalization condition on $U_{21}(r)$ is

$$1 = \int_0^\infty |U_{21}(r)|^2 dr =$$

Special integral: $\int_0^\infty r^4 e^{-r/a_0} dr = (4!) a_0^5$ (Dwight 860.07)

Find $C_0 =$ $= \sqrt{\frac{2}{3a_0}}$

Finally $\Psi_{210}(r, \theta, \varphi) =$

$$\Psi_{210}(r, \theta, \varphi) = \sqrt{\frac{1}{32\pi a_0^2}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \cos\theta$$

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QM QUESTION - Why construct $L_\pm = L_x \pm i L_y$?

Recall that for QHO we constructed $a_\pm = A(\mp ip + m\omega x)$
because we sought eigenstates & eigenvalues for

$$H\psi = \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right)\psi$$ Griffiths QM p. 42

$$E\psi = \frac{1}{2m}\left(p^2 + (m\omega x)^2\right)\psi$$

We'd like to be able to factor $H = \frac{1}{2m}\left(p^2 + (m\omega x)^2\right)$
$u^2 + v^2$

If p & x weren't operators, we could just use $u^2+v^2 = (iu+v)(-iu+v)$

This motivates us to construct $a_\pm = A(\mp ip + m\omega x)$...
$[x,p] = i\hbar$

Similarly, when seeking simultaneous eigenstates of the angular momentum operators L^2 and L_z

Shankar p. 330 (1) $L^2|\alpha\beta\rangle = \alpha|\alpha\beta\rangle$ and $L_z|\alpha\beta\rangle = \beta|\alpha\beta\rangle$

value state value state

Use $L^2 = L_z^2 + L_x^2 + L_y^2$ to write

$$(L^2 - L_z^2)|\alpha\beta\rangle = (\alpha - \beta)|\alpha\beta\rangle = (L_x^2 + L_y^2)|\alpha\beta\rangle$$
$$= u^2 + v^2$$

By analogy with $a_\pm$,

(12.5.3) This motivates us to construct $L_\pm = L_x + iL_y$ ...

$[L_x, L_y] = i\hbar L_z$