PS wk 3 17 Apr 07

Shankar QM Ch 7 - Harmonic Oscillator

$-\frac{dV}{dx} = F = m\ddot{x} = -kx = -m\omega^2 x \Rightarrow \omega^2 = \frac{k}{m}$, $V = \frac{1}{2}kx^2$

$V = \frac{1}{2}m\omega^2 x^2$

$\mathcal{H} = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

QHO: Solution to $H\psi_E = E\psi_E \rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{m\omega^2 x^2}{2}\right)\psi = 0$

① Let $x = by \rightarrow \frac{d^2\psi}{dy^2} + \frac{2mEb^2}{\hbar^2}\psi - \frac{m^2\omega^2 b^4}{\hbar^2}y^2\psi = 0 \rightarrow \psi'' + (2\epsilon - y^2)\psi = 0$

$b^2 = \frac{\hbar}{m\omega}$ $\quad\epsilon = \frac{E}{\hbar\omega}$

② $y \to \infty$: $\psi'' = y^2\psi \rightarrow \psi = y^m e^{-y^2/2}$

$y \to 0$: $\psi'' = -2\epsilon\psi \rightarrow \psi \to A + cy$

$\psi = u(y)e^{-y^2/2}$

③ $u'' - 2yu' + (2\epsilon - 1)u = 0 \rightarrow u(y) = \sum_{n=0}^{\infty} C_n y^n$, $\frac{C_{n+2}}{C_n} = \frac{(2n+1-2\epsilon)}{(n+2)(n+1)}$

④ $\epsilon_n = \frac{2n+1}{2} = n + \frac{1}{2}$, $n = 0, 1, 2 \cdots$ $\quad E_n = (n + \frac{1}{2})\hbar\omega$

$\psi_E(x) = A_n e^{\frac{-m\omega x^2}{2\hbar}} H_n(x)$ where $A_n^4 = \frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}$, H_n = Hermite polynomials

Raising & lowering operators

7.4 QHO in Energy Basis : a^+ and a

$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} X + i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} P$, $a^+|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$

$a^+ = \quad " \quad - \quad "$, $a|n\rangle = n^{1/2}\,|n-1\rangle$

$[a, a^+] = 1 \qquad a^+a = \frac{H}{\hbar\omega} - \frac{1}{2} \qquad H = (a^+a + \frac{1}{2})\hbar\omega = P^2 + X^2$

$a|\varepsilon_0\rangle = 0 \qquad a^+a|\varepsilon_0\rangle = 0 \qquad H|\varepsilon_0\rangle = \frac{1}{2}|\varepsilon_0\rangle$

$\hat{H}\,|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$

number operator $\quad \hat{H} = N + \frac{1}{2}$

$X = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a + a^+) \qquad P = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a^+ - a)$

Ex : $\langle 3|X^3|2\rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2}\langle 3|(a+a^+)^3|2\rangle$ 220

matrix element of X^3 operator between $\langle 3|$ and $|2\rangle$ eigen states

7.5 Energy basis → X basis

Amplitude for finding particle in a state $|n\rangle$ at the point x

$\psi_n(x) = \langle x|n\rangle \qquad |n\rangle = \frac{(a^+)^n}{(n!)^{1/2}}|0\rangle$

$a = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right) \qquad a^+ = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right) \qquad y = \sqrt{\frac{m\omega}{\hbar}}\,x$

$\langle x|n\rangle = \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)\right]^n \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-y^2/2}$, $H_n(y) = e^{y^2/2}\left(y - \frac{d}{dy}\right)^n e^{-y^2/2}$