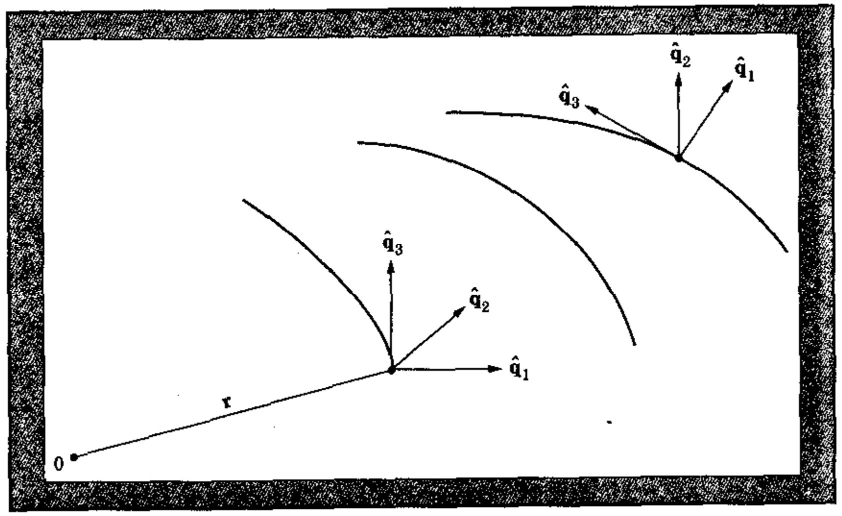
ARFKEN & WEBER - Essential Math Methods for Physicists - 2004

Figure 2.14

Curved Coordinates q_i with Varying Directions $\hat{\mathbf{q}}_i$



radial and angular components. A relevant example is the motion of a planet around a central star in plane polar coordinates (Example 2.2.5). In other words, we can again break vectors into components efficiently: $(x, y) \rightarrow (\rho, \varphi)$, a powerful concept of physics and engineering. In this section, we develop the general formalism of orthogonal coordinates, **derive from the geometry of orthogonal coordinates the coordinate differentials, and use them for line, area, and volume elements in multiple integrals.**

We may describe any point (x, y, z) as the intersection of three planes in Cartesian coordinates or as the intersection of the three surfaces that form our new, curvilinear coordinates as sketched in Fig. 2.14. Describing the curvilinear coordinate surfaces by q_1 = constant, q_2 = constant, q_3 = constant, we may identify our point by (q_1, q_2, q_3) as well as by (x, y, z). This means that in principle we may write

$$\begin{array}{ll} \text{General curvilinear coordinates} & \text{Circular cylindrical coordinates} \\ q_1, q_2, q_3 & \rho, \varphi, z \\ x = x(q_1, q_2, q_3) & -\infty < x = \rho\cos\varphi < \infty \\ y = y(q_1, q_2, q_3) & -\infty < y = \rho\sin\varphi < \infty \\ z = z(q_1, q_2, q_3) & -\infty < z = z < \infty \end{array} \tag{2.27}$$

specifying x, y, z in terms of the q's and the inverse relations,

$$\begin{array}{ll} q_1 = q_1(x, y, z) & 0 \le \rho = (x^2 + y^2)^{1/2} < \infty \\ q_2 = q_2(x, y, z) & 0 \le \varphi = \arctan(y/x) < 2\pi \\ q_3 = q_3(x, y, z) & -\infty < z = z < \infty. \end{array} \tag{2.28}$$

Derivation of Differential vector operators in any coordinate system

As a specific illustration of the general, abstract q_1, q_2, q_3, the transformation equations for circular cylindrical coordinates (Section 2.2) are included in Eqs. (2.27) and (2.28). With each family of surfaces q_i = constant, we can associate a unit vector $\hat{\mathbf{q}}_i$ normal to the surface q_i = constant and in the direction of increasing q_i. Because the normal to the q_i = constant surfaces can point in different directions depending on the position in space (remember that these surfaces are not planes), **the unit vectors $\hat{\mathbf{q}}_i$ can depend on the position in space,** just like $\hat{\varphi}$ in cylindrical coordinates. Then the coordinate vector and a vector $\mathbf{V}$ may be written as

$$\mathbf{r} = \hat{\mathbf{q}}_1 q_1 + \hat{\mathbf{q}}_2 q_2 + \hat{\mathbf{q}}_3 q_3, \quad \mathbf{V} = \hat{\mathbf{q}}_1 V_1 + \hat{\mathbf{q}}_2 V_2 + \hat{\mathbf{q}}_3 V_3. \tag{2.29}$$

The $\hat{\mathbf{q}}_i$ are normalized to $\hat{\mathbf{q}}_i^2 = 1$ and form a right-handed coordinate system with volume $\hat{\mathbf{q}}_1 \cdot (\hat{\mathbf{q}}_2 \times \hat{\mathbf{q}}_3) = 1$.

This example tells us that we need to differentiate $x(q_1, q_2, q_3)$ in Eq. (2.27), and this leads to (see total differential in Section 1.5)

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3, \tag{2.30}$$

and similarly for differentiation of y and z, that is, $d\mathbf{r} = \sum_i \frac{\partial \mathbf{r}}{\partial q_i} dq_i$.

In curvilinear coordinate space the most general expression for the square of the distance element can be written as a quadratic form:

$$\begin{aligned} ds^2 &= g_{11}\,dq_1^2 + g_{12}\,dq_1\,dq_2 + g_{13}\,dq_1\,dq_3 \\ &\quad + g_{21}\,dq_2\,dq_1 + g_{22}\,dq_2^2 + g_{23}\,dq_2\,dq_3 \\ &\quad + g_{31}\,dq_3\,dq_1 + g_{32}\,dq_3\,dq_2 + g_{33}\,dq_3^2 \\ &= \sum_{ij} g_{ij}\,dq_i\,dq_j, \end{aligned} \tag{2.31}$$

where the mixed terms $dq_i dq_j$ with $i \neq j$, signal that these coordinates are not orthogonal. Spaces for which Eq. (2.31) is the definition of distance are called metric and Riemannian. Substituting Eq. (2.30) (squared) and the corresponding results for dy^2 and dz^2 into Eq. (2.2) and equating coefficients of $dq_i\,dq_j$,[2] we find

$$g_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} = \sum_l \frac{\partial x_l}{\partial q_i}\frac{\partial x_l}{\partial q_j}. \tag{2.32}$$

These coefficients g_{ij}, which we now proceed to investigate, may be viewed as specifying the nature of the coordinate system (q_1, q_2, q_3). Collectively, these coefficients are referred to as the **metric**.[3] In general relativity the metric components are determined by the properties of matter, that is, the g_{ij} are solutions of Einstein's nonlinear field equations that are driven by the energy-momentum tensor of matter: Geometry is merged with physics.

[2] The dq are arbitrary. For instance, setting $dq_2 = dq_3 = 0$ isolates g_{11}. Note that Eq. (2.32) can be derived from Eq. (2.30) more elegantly with the matrix notation of Chapter 3.

[3] The tensor nature of the set of g_{ij} follows from the quotient rule (Section 2.8). Then the tensor transformation law yields Eq. (2.32).

From this point on, we limit ourselves to **orthogonal** coordinate systems (defined by mutually **perpendicular** surfaces or, equivalently, sums of squares in ds^2),[4] which means (Exercise 2.3.1)

$$g_{ij} = 0, \qquad i \neq j, \quad \text{or} \quad \hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_j = \delta_{ij}. \tag{2.33}$$

Now, to simplify the notation, we write $g_{ii} = h_i^2$ so that

$$ds^2 = (h_1\,dq_1)^2 + (h_2\,dq_2)^2 + (h_3\,dq_3)^2 = \sum_i (h_i\,dq_i)^2. \tag{2.34}$$

The specific orthogonal coordinate systems in Sections 2.2 and 2.5 are described by specifying **scale factors** h_1, h_2, and h_3. Conversely, the scale factors may be conveniently identified by the relation

$$ds_i = h_i\,dq_i \tag{2.35}$$

for any given dq_i, holding the other q's constant. Note that the three curvilinear coordinates q_1, q_2, q_3 need not be lengths. The scale factors h_i may depend on the q's and they may have dimensions. The **product** $h_i\,dq_i$ must have dimensions of length and be positive. Because Eq. (2.32) can also be written as a scalar product of the **tangent vectors**

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j}, \tag{2.36}$$

the orthogonality condition in Eq. (2.33) in conjunction with the sum of squares in Eq. (2.34) tell us that for each displacement along a coordinate axis (see Fig. 2.14)

$$\frac{\partial \mathbf{r}}{\partial q_i} = h_i \hat{\mathbf{q}}_i, \tag{2.37}$$

they are the coordinate tangent vectors so that the differential distance vector $d\mathbf{r}$ becomes

$$d\mathbf{r} = \sum_i h_i\,dq_i\,\hat{\mathbf{q}}_i = \sum_i h_i d\mathbf{q}_i. \tag{2.38}$$

Using the curvilinear component form we find that a **line integral** becomes

$$\int \mathbf{V} \cdot d\mathbf{r} = \sum_i \int V_i h_i\,dq_i. \tag{2.39}$$

The work $dW = \mathbf{F} \cdot d\mathbf{r}$ done by a force $\mathbf{F}$ along a line element $d\mathbf{r}$ is the most prominent example in physics for a line integral. In this context, we often use the **chain rule** in the form

$$\int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathbf{A}(\mathbf{r(t)}) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{A}(\mathbf{r(t)}) \cdot \frac{d\mathbf{r}}{dt} dt. \tag{2.40}$$

[4] In relativistic cosmology the nondiagonal elements of the metric g_{ij} are usually set equal to zero as a consequence of physical assumptions such as no rotation.

EXAMPLE 2.3.1

Energy Conservation for Conservative Force Using Eq. (2.40) for a force $\mathbf{F} = m\frac{d\mathbf{v}}{dt}$ in conjunction with Newton's equation of motion for a particle of mass m allows us to integrate analytically the work

$$\int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathbf{F} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}(t)}{dt} dt = m \int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt = \frac{m}{2} \int \frac{d\mathbf{v}^2}{dt} dt = \frac{m}{2} \mathbf{v}^2 |_{t_1}^{t_2}$$
$$= \frac{m}{2} [\mathbf{v}^2(t_2) - \mathbf{v}^2(t_1)]$$

as the difference of kinetic energies. If the force derives from a potential as $\mathbf{F} = -\nabla V$, then we can **integrate that line integral explicitly because it contains the gradient**

$$\int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \nabla V(\mathbf{r}) \cdot d\mathbf{r} = -V|_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} = -[V(\mathbf{r}(t_2)) - V(\mathbf{r}(t_1))]$$

and identify the work as minus the potential difference. Comparing both expressions, we have **energy conservation**

$$\frac{m}{2}\mathbf{v}^2(t_2) + V(\mathbf{r}(t_2)) = \frac{m}{2}\mathbf{v}^2(t_1) + V(\mathbf{r}(t_1))$$

for a conservative force. The path independence of the work is discussed in detail in Section 1.12. Thus, in this case only the end points of the path $\mathbf{r}(t)$ matter. ■

In Cartesian coordinates the **length of a space curve** is given by $\int ds$, with $ds^2 = dx^2 + dy^2 + dz^2$. If a space curve in curved coordinates is parameterized as $(q_1(t),\ q_2(t),\ q_3(t))$, we find its length by integrating the length element of Eq. (2.34) so that

$$L = \int_{t_1}^{t_2} \sqrt{h_1^2 \left(\frac{dq_1}{dt}\right)^2 + h_2^2 \left(\frac{dq_2}{dt}\right)^2 + h_3^2 \left(\frac{dq_3}{dt}\right)^2}\, dt \tag{2.41}$$

using the chain rule [Eq. (2.40)]. From Eq. (2.35) we immediately develop the **area and volume elements**

$$d\sigma_{ij} = ds_i\,ds_j = h_i h_j\,dq_i\,dq_j \tag{2.42}$$

and

$$d\tau = ds_1\,ds_2\,ds_3 = h_1 h_2 h_3\,dq_1\,dq_2\,dq_3. \tag{2.43}$$

From Eq. (2.42) an area element may be expanded:

$$\begin{aligned} d\boldsymbol{\sigma} &= ds_2\,ds_3\,\hat{\mathbf{q}}_1 + ds_3\,ds_1\,\hat{\mathbf{q}}_2 + ds_1\,ds_2\,\hat{\mathbf{q}}_3 \\ &= h_2 h_3\,dq_2\,dq_3\,\hat{\mathbf{q}}_1 + h_3 h_1\,dq_3\,dq_1\,\hat{\mathbf{q}}_2 \\ &\quad + h_1 h_2\,dq_1\,dq_2\,\hat{\mathbf{q}}_3. \end{aligned} \tag{2.44}$$

Thus, a **surface integral** becomes

$$\int \mathbf{V}\cdot d\sigma = \int V_1 h_2 h_3\, dq_2\, dq_3 + \int V_2 h_3 h_1\, dq_3\, dq_1 + \int V_3 h_1 h_2\, dq_1\, dq_2. \tag{2.45}$$

More examples of such line and surface integrals in cylindrical and spherical polar coordinates appear in Sections 2.2 and 2.5.

In anticipation of the new forms of equations for vector **calculus** that appear in the next section, we emphasize that vector **algebra** is the same in orthogonal curvilinear coordinates as in Cartesian coordinates. Specifically, for the dot product

$$\mathbf{A}\cdot\mathbf{B} = \sum_{i,k} A_i \hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_k B_k = \sum_{i,k} A_i B_k \delta_{ik} = \sum_i A_i B_i, \tag{2.46}$$

where the subscripts indicate curvilinear components. For the cross product

$$\mathbf{A}\times\mathbf{B} = \sum_{i,k} A_i \hat{\mathbf{q}}_i \times \hat{\mathbf{q}}_k B_k = \begin{vmatrix} \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 & \hat{\mathbf{q}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}, \tag{2.47}$$

as in Eq. (1.40).

EXAMPLE 2.3.2

Orbital Angular Momentum in Cylindrical Coordinates In circular cylindrical coordinates the orbital angular momentum takes the form [see Eq. (2.8)] for the coordinate vector $\mathbf{r} = \boldsymbol{\rho} + \mathbf{z}$ and Example 2.2.5 for the velocity $\mathbf{v} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{\mathbf{z}}$

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{\mathbf{z}} \\ \rho & 0 & z \\ \dot{\rho} & \rho\dot{\varphi} & \dot{z} \end{vmatrix}. \tag{2.48}$$

Now let us take the mass to be 3 kg, the lever arm as 1 m in the radial direction of the xy-plane, and the velocity as 2 m/s in the z-direction. Then we expect $\mathbf{L}$ to be in the $\hat{\varphi}$ direction and quantitatively

$$\mathbf{L} = 3\begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{\mathbf{z}} \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 3\hat{\rho}\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} - 3\hat{\varphi}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + 3\hat{\mathbf{z}}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = -6\hat{\varphi}\,\mathrm{mkg/s}. \tag{2.49}$$

■

Previously, we specialized to locally rectangular coordinates that are adapted to special symmetries. Let us now briefly examine the more general case in which the coordinates are not necessarily orthogonal. Surface and volume elements are part of multiple integrals, which are common in physical applications such as center of mass determinations and moments of inertia. Typically, we choose coordinates according to the symmetry of the particular problem. In Chapter 1 we used Gauss's theorem to transform a volume integral into a surface integral and Stokes's theorem to transform a surface integral into a line integral. For orthogonal coordinates, the surface and volume elements are simply products of the line elements $h_i dq_i$ [see Eqs. (2.42) and (2.43)]. For the general case, we use the geometric meaning of $\partial\mathbf{r}/\partial q_i$ in Eq. (2.37) as tangent vectors. We start with the Cartesian surface element $dx\,dy$, which becomes an infinitesimal rectangle in the new coordinates q_1, q_2 formed by the two incremental vectors

$$d\mathbf{r}_1 = \mathbf{r}(q_1 + dq_1, q_2) - \mathbf{r}(q_1, q_2) = \frac{\partial\mathbf{r}}{\partial q_1} dq_1,$$
$$d\mathbf{r}_2 = \mathbf{r}(q_1, q_2 + dq_2) - \mathbf{r}(q_1, q_2) = \frac{\partial\mathbf{r}}{\partial q_2} dq_2, \tag{2.50}$$

whose area is the z-component of their cross product, or

$$dx\,dy = d\mathbf{r}_1 \times d\mathbf{r}_2\big|_z = \left[\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}\right] dq_1\,dq_2$$
$$= \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{vmatrix} dq_1\,dq_2. \tag{2.51}$$

The transformation coefficient in determinant form is called the **Jacobian**.

Similarly, the volume element $dx\,dy\,dz$ becomes the triple scalar product of the three infinitesimal displacement vectors $d\mathbf{r}_i = dq_i \frac{\partial\mathbf{r}}{\partial q_i}$ along the q_i directions $\hat{\mathbf{q}}_i$, which according to Section 1.4 takes on the form

$$dx\,dy\,dz = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix} dq_1\,dq_2\,dq_3. \tag{2.52}$$

Here, the coefficient is also called the Jacobian, and so on in higher dimensions.

For orthogonal coordinates the Jacobians simplify to products of the orthogonal vectors in Eq. (2.38). It follows that they are products of h_i; for example, the volume Jacobian in Eq. (2.52) becomes

$$h_1 h_2 h_3 (\hat{\mathbf{q}}_1 \times \hat{\mathbf{q}}_2)\cdot\hat{\mathbf{q}}_3 = h_1 h_2 h_3.$$

EXAMPLE 2.3.3

Jacobians for Polar Coordinates Let us illustrate the transformation of the Cartesian two-dimensional volume element $dx\,dy$ to polar coordinates ρ, φ, with $x = \rho\cos\varphi,\ y = \rho\sin\varphi$. (See also Section 2.2.) Here,

$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{vmatrix} d\rho\,d\varphi = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} d\rho\,d\varphi = \rho\,d\rho\,d\varphi. \tag{2.53}$$

Similarly, in spherical coordinates (see Section 2.5), we get from $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$ the Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix}$$

$$= r^2(\cos^2\theta\sin\theta + \sin^3\theta) = r^2\sin\theta \tag{2.54}$$

by expanding the determinant along the third line. Hence, the volume element becomes $dx\,dy\,dz = r^2 dr\sin\theta d\theta d\varphi$. The volume integral can be written as

$$\int f(x, y, z)dx\,dy\,dz = \int f(x(r, \theta, \varphi), y(r, \theta, \varphi), z(r, \theta, \varphi))r^2 dr\sin\theta d\theta d\varphi. \quad \blacksquare$$

SUMMARY

We have developed the general formalism for vector analysis in curvilinear coordinates. For most applications, locally orthogonal coordinates can be chosen, for which surface and volume elements in multiple integrals are products of line elements. For the general nonorthogonal case, Jacobian determinants apply.

Biographical Data

Jacobi, Carl Gustav Jacob. Jacobi, a German mathematician, was born in Potsdam, Prussia, in 1804 and died in Berlin in 1851. He obtained his Ph.D. in Berlin in 1824. Praised by Legendre for his work on elliptical functions, he became a professor at the University of Königsberg in 1827 (East Prussia, which is now Russia). He also developed determinants and partial differential equations, among other contributions.

EXERCISES

2.3.1 Show that limiting our attention to orthogonal coordinate systems implies that $g_{ij} = 0$ for $i \neq j$ [Eq. (2.33)].
Hint. Construct a triangle with sides ds_1, ds_2, and ds_3. Equation (2.42) must hold regardless of whether $g_{ij} = 0$. Then compare ds^2 from Eq. (2.34) with a calculation using the law of cosines. Show that $\cos\theta_{12} = g_{12}/\sqrt{g_{11}g_{22}}$.

2.3.2 In the spherical polar coordinate system $q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$. The transformation equations corresponding to Eq. (2.27) are

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

(a) Calculate the spherical polar coordinate scale factors: h_r, h_θ, and h_φ.
(b) Check your calculated scale factors by the relation $ds_i = h_i\,dq_i$.

2.3.3 The u-, v-, z-coordinate system frequently used in electrostatics and in hydrodynamics is defined by

$$xy = u, \quad x^2 - y^2 = v, \quad z = z.$$

This u-, v-, z-system is orthogonal.
(a) In words, describe briefly the nature of each of the three families of coordinate surfaces.
(b) Sketch the system in the xy-plane showing the intersections of surfaces of constant u and surfaces of constant v with the xy-plane (using graphical software if available).
(c) Indicate the directions of the unit vector $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in all four quadrants.
(d) Is this u-, v-, z-system right-handed ($\hat{\mathbf{u}} \times \hat{\mathbf{v}} = +\hat{\mathbf{z}}$) or left-handed ($\hat{\mathbf{u}} \times \hat{\mathbf{v}} = -\hat{\mathbf{z}}$)?

2.3.4 The elliptic cylindrical coordinate system consists of three families of surfaces:

$$\frac{x^2}{a^2\cosh^2 u} + \frac{y^2}{a^2\sinh^2 u} = 1, \qquad \frac{x^2}{a^2\cos^2 v} - \frac{y^2}{a^2\sin^2 v} = 1, \qquad z = z.$$

Sketch the coordinate surfaces u = constant and v = constant as they intersect the first quadrant of the xy-plane (using graphical software if available). Show the unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. The range of u is $0 \le u < \infty$, and the range of v is $0 \le v \le 2\pi$.
Hint. It is easier to work with the square of each side of this equation.

2.3.5 Determine the volume of an n-dimensional sphere of radius r.
Hint. Use generalized polar coordinates.

2.3.6 Minkowski space is defined as $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_0 = ct$. This is done so that the space–time interval $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ (c = velocity of light). Show that the metric in Minkowski space is

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We use Minkowski space in Section 4.4 for describing Lorentz transformations.

2.4 Differential Vector Operators

Gradient

The starting point for developing the gradient, divergence, and curl operators in curvilinear coordinates is the geometric interpretation of the gradient as the vector having the magnitude and direction of the maximum space rate of change of a function ψ (compare Section 1.5). From this interpretation the component of $\nabla\psi(q_1, q_2, q_3)$ in the direction normal to the family of surfaces $q_1 = \text{constant}$ is given by[5]

$$\hat{\mathbf{q}}_1 \cdot \nabla\psi = \nabla\psi|_1 = \frac{\partial\psi}{\partial s_1} = \frac{1}{h_1}\frac{\partial\psi}{\partial q_1} \tag{2.55}$$

since this is the rate of change of ψ for varying q_1, holding q_2 and q_3 fixed. For example, from Example 2.2.1 [and $h_\rho = 1$, $h_\varphi = \rho$, $h_z = 1$ from Eq. (2.9)] the φ-component of the gradient in circular cylindrical coordinates has the form given by Eq. (2.15). The quantity ds_1 is a differential length in the direction of increasing q_1 [compare Eq. (2.35)]. In Section 2.3, we introduced a unit vector $\hat{\mathbf{q}}_1$ to indicate this direction. By repeating Eq. (2.55) for q_2 and again for q_3 and adding vectorially, the gradient becomes

$$\begin{aligned}\nabla\psi(q_1, q_2, q_3) &= \hat{\mathbf{q}}_1\frac{\partial\psi}{\partial s_1} + \hat{\mathbf{q}}_2\frac{\partial\psi}{\partial s_2} + \hat{\mathbf{q}}_3\frac{\partial\psi}{\partial s_3} \\ &= \hat{\mathbf{q}}_1\frac{1}{h_1}\frac{\partial\psi}{\partial q_1} + \hat{\mathbf{q}}_2\frac{1}{h_2}\frac{\partial\psi}{\partial q_2} + \hat{\mathbf{q}}_3\frac{1}{h_3}\frac{\partial\psi}{\partial q_3} \\ &= \sum_i \hat{\mathbf{q}}_i\frac{1}{h_i}\frac{\partial\psi}{\partial q_i}.\end{aligned} \tag{2.56}$$

Exercise 2.2.4 offers a mathematical alternative independent of this physical interpretation of the gradient. Examples are given for cylindrical coordinates in Section 2.2 and spherical polar coordinates in Section 2.5.

Divergence

The divergence operator may be obtained from the second definition [Eq. (1.113)] of Chapter 1 or equivalently from Gauss's theorem (Section 1.11). Let us use Eq. (1.113),

$$\nabla\cdot\mathbf{V}(q_1, q_2, q_3) = \lim_{\int d\tau\to 0}\frac{\int \mathbf{V}\cdot d\sigma}{\int d\tau}, \tag{2.57}$$

with a differential volume $d\tau = h_1h_2h_3\,dq_1\,dq_2\,dq_3$ (Fig. 2.15). Note that the positive directions have been chosen so that (q_1, q_2, q_3) or $(\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2, \hat{\mathbf{q}}_3)$ form a right-handed set, $\hat{\mathbf{q}}_1 \times \hat{\mathbf{q}}_2 = \hat{\mathbf{q}}_3$.

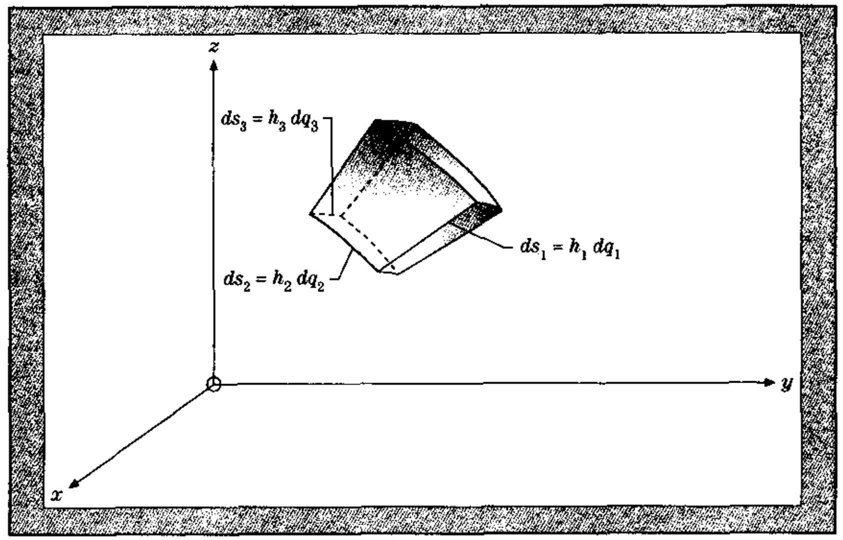
The area integral for the two faces $q_1 = \text{constant}$ in Fig. 2.15 is given by

$$\begin{aligned}&\left[V_1h_2h_3 + \frac{\partial}{\partial q_1}(V_1h_2h_3)dq_1\right]dq_2\,dq_3 - V_1h_2h_3\,dq_2\,dq_3 \\ &\quad = \frac{\partial}{\partial q_1}(V_1h_2h_3)dq_1\,dq_2\,dq_3,\end{aligned} \tag{2.58}$$

[5]Here, the use of φ to label a function is avoided because it is conventional to use this symbol to denote an azimuthal coordinate.

Figure 2.15

Curvilinear Volume Element



as in Sections 1.6 and 1.9.[6] Here, V_i is the component of $\mathbf{V}$ in the $\hat{\mathbf{q}}_i$-direction, increasing q_i; that is, $V_i = \hat{\mathbf{q}}_i\cdot\mathbf{V}$ is the projection of $\mathbf{V}$ onto the $\hat{\mathbf{q}}_i$-direction. Adding in the similar results for the other two pairs of surfaces, we obtain

$$\int \mathbf{V}(q_1, q_2, q_3)\cdot d\sigma = \left[\frac{\partial}{\partial q_1}(V_1h_2h_3) + \frac{\partial}{\partial q_2}(V_2h_3h_1) + \frac{\partial}{\partial q_3}(V_3h_1h_2)\right]dq_1\,dq_2\,dq_3. \tag{2.59}$$

Division by our differential volume [see $d\tau$ after Eq. (2.57)] yields

$$\nabla\cdot\mathbf{V}(q_1, q_2, q_3) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(V_1h_2h_3) + \frac{\partial}{\partial q_2}(V_2h_3h_1) + \frac{\partial}{\partial q_3}(V_3h_1h_2)\right]. \tag{2.60}$$

Applications and examples of this general result will be given in the following section for a special coordinate system. We may obtain the Laplacian by combining Eqs. (2.56) and (2.60), using $\mathbf{V} = \nabla\psi(q_1, q_2, q_3)$. This leads to

$$\begin{aligned}\nabla\cdot\nabla\psi(q_1, q_2, q_3) = \frac{1}{h_1h_2h_3}&\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right)\right. \\ &\left. + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right].\end{aligned} \tag{2.61}$$

Examples and numerical applications of the central Eqs. (2.56), (2.60), (2.61), and (2.66) are shown for cylindrical coordinates in Section 2.2 and for spherical polar coordinates in Section 2.5.

[6]Since we take the limit $dq_1, dq_2, dq_3 \to 0$, the second- and higher order derivatives will drop out.

Curl

Finally, to develop $\nabla \times \mathbf{V}$, let us apply Stokes's theorem (Section 1.11) and, as with divergence, take the limit as the surface area becomes vanishingly small. Working on one component at a time, we consider a differential surface element in the curvilinear surface $q_1 =$ constant. For such a small surface the mean value theorem of integral calculus states that an integral is given by the surface times the function at a mean value on the small surface. Thus, from

$$\int_s \nabla \times \mathbf{V}|_1 \cdot d\sigma_1 = \hat{\mathbf{q}}_1 \cdot (\nabla \times \mathbf{V}) h_2 h_3 \, dq_2 \, dq_3, \tag{2.62}$$

Stokes's theorem yields

$$\hat{\mathbf{q}}_1 \cdot (\nabla \times \mathbf{V}) h_2 h_3 \, dq_2 \, dq_3 = \oint \mathbf{V} \cdot d\mathbf{r}, \tag{2.63}$$

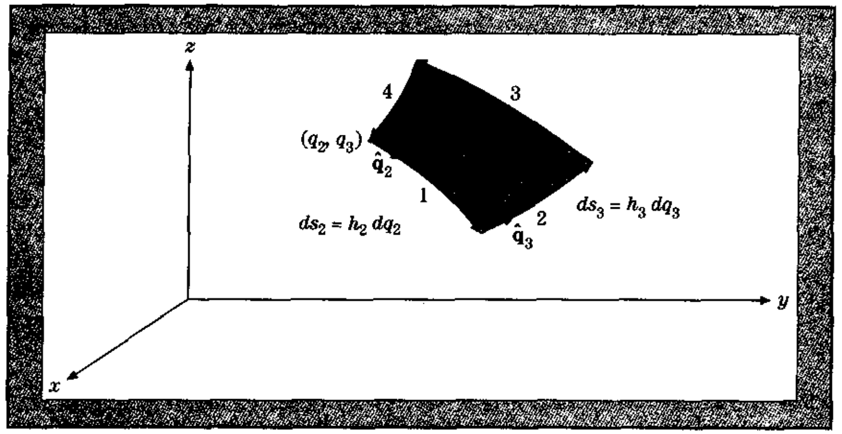
with the line integral lying in the surface $q_1 =$ constant. Following the loop (1, 2, 3, 4) of Fig. 2.16,

$$\begin{aligned} \oint \mathbf{V}(q_1, q_2, q_3) \cdot d\mathbf{r} &= V_2 h_2 \, dq_2 + \left[V_3 h_3 + \frac{\partial}{\partial q_2}(V_3 h_3)\, dq_2 \right] dq_3 \\ &\quad - \left[V_2 h_2 + \frac{\partial}{\partial q_3}(V_2 h_2) dq_3 \right] dq_2 - V_3 h_3 \, dq_3 \\ &= \left[\frac{\partial}{\partial q_2}(h_3 V_3) - \frac{\partial}{\partial q_3}(h_2 V_2) \right] dq_2 \, dq_3. \end{aligned} \tag{2.64}$$

We pick up a positive sign when going in the positive direction on parts 1 and 2 and a negative sign on parts 3 and 4 because here we are going in the negative direction. Higher order terms have been omitted. They will vanish in the limit as the surface becomes vanishingly small ($dq_2 \to 0$, $dq_3 \to 0$).

Figure 2.16

Curvilinear Surface Element with $q_1 =$ Constant



Combining Eqs. (2.63) and (2.64), we obtain

$$\nabla \times \mathbf{V}|_1 = \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial q_2}(h_3 V_3) - \frac{\partial}{\partial q_3}(h_2 V_2) \right]. \tag{2.65}$$

The remaining two components of $\nabla \times \mathbf{V}$ may be picked up by cyclic permutation of the indices. As in Chapter 1, it is often convenient to write the curl in determinant form:

$$\nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\mathbf{q}}_1 h_1 & \hat{\mathbf{q}}_2 h_2 & \hat{\mathbf{q}}_3 h_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}. \tag{2.66}$$

Remember that because of the presence of the differential operators, this determinant must be expanded from the top down. Note that this equation is **not** identical to the form for the cross product of two vectors [Eq. (1.40)]. ∇ is not an ordinary vector; it is a vector **operator**.

Our geometric interpretation of the gradient and the use of Gauss's and Stokes's theorems (or integral definitions of divergence and curl) have enabled us to obtain these general formulas **without having to differentiate the unit vectors $\hat{\mathbf{q}}_i$**. There exist alternate ways to determine grad, div, and curl based on direct differentiation of the $\hat{\mathbf{q}}_i$. One approach resolves the $\hat{\mathbf{q}}_i$ of a specific coordinate system into its Cartesian components (Exercises 2.2.1 and 2.5.1) and differentiates this Cartesian form (Exercises 2.4.3 and 2.5.2). The point is that the derivatives of the Cartesian $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ vanish since $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are constant in direction as well as in magnitude. A second approach [L. J. Kijewski, *Am. J. Phys.* **33**, 816 (1965)] starts from the equality of $\partial^2 \mathbf{r}/\partial q_i \, \partial q_j$ and $\partial^2 \mathbf{r}/\partial q_j \, \partial q_i$ and develops the derivatives of $\hat{\mathbf{q}}_i$ in a general curvilinear form. Exercises 2.3.3 and 2.3.4 are based on this method.

EXERCISES

2.4.1 Develop arguments to show that ordinary dot and cross products (not involving ∇) in orthogonal curvilinear coordinates proceed as in Cartesian coordinates **with no involvement of scale factors**.

2.4.2 With $\hat{\mathbf{q}}_1$ a unit vector in the direction of increasing q_1, show that

(a) $\nabla \cdot \hat{\mathbf{q}}_1 = \dfrac{1}{h_1 h_2 h_3} \dfrac{\partial (h_2 h_3)}{\partial q_1}$

(b) $\nabla \times \hat{\mathbf{q}}_1 = \dfrac{1}{h_1} \left[\hat{\mathbf{q}}_2 \dfrac{1}{h_3} \dfrac{\partial h_1}{\partial q_3} - \hat{\mathbf{q}}_3 \dfrac{1}{h_2} \dfrac{\partial h_1}{\partial q_2} \right]$.

Note that even though $\hat{\mathbf{q}}_1$ is a unit vector, its divergence and curl **do not necessarily vanish** (because it varies with position).