so

$$\frac{C_{n+2}}{C_n} \xrightarrow[n\to\infty]{} \frac{[(n-m)/2]!}{[(n+2-m)/2]!} = \frac{1}{(n-m+2)/2} \sim \frac{2}{n}$$

In other words, $u(y)$ in Eq. (7.3.16) grows as $y^m e^{y^2}$, so that $\psi(y) \simeq y^m e^{y^2} e^{-y^2/2} \simeq y^m e^{+y^2/2}$, which is the rejected solution raising its ugly head! Our predicament is now reversed: from finding that every ε is allowed, we are now led to conclude that no ε is allowed. Fortunately there is a way out. If ε is one of the special values

$$\varepsilon_n = \frac{2n+1}{2}, \quad n = 0, 1, 2, \ldots \tag{7.3.18}$$

the coefficient C_{n+2} (and others dependent on it) vanish. If we choose $C_1 = 0$ when n is even (or $C_0 = 0$ when n is odd) we have a finite polynomial of order n which satisfies the differential equation and behaves as y^n as $y \to \infty$:

$$\psi(y) = u(y)e^{-y^2/2} = \left\{\begin{matrix} C_0 + C_2y^2 + C_4y^4 + \cdots + C_ny^n \\ \text{or} \\ C_1y + C_3y^3 + C_5y^5 + \cdots + C_ny^n \end{matrix}\right\} \cdot e^{-y^2/2} \tag{7.3.19}$$

Equation (7.3.18) tells us that energy is quantized: the only allowed values for $E = \varepsilon\hbar\omega$ (that is, values that yield solutions in the physical Hilbert space) are

$$E_n = (n + \tfrac{1}{2})\hbar\omega, \quad n = 0, 1, 2, \ldots \tag{7.3.20}$$

For each value of n, Eq. (7.3.15) determines the corresponding polynomials of nth order, called *Hermite polynomials*, $H_n(y)$:

$$\begin{aligned} H_0(y) &= 1 \\ H_1(y) &= 2y \\ H_2(y) &= -2(1 - 2y^2) \\ H_3(y) &= -12(y - \tfrac{2}{3}y^3) \\ H_4(y) &= 12(1 - 4y^2 + \tfrac{4}{3}y^4) \end{aligned} \tag{7.3.21}$$

The arbitrary initial coefficients C_0 and C_1 in H_n are chosen according to a standard convention. The normalized solutions are then

$$\begin{aligned} \psi_E(x) &= \psi_{(n+1/2)\hbar\omega}(x) = \psi_n(x) \\ &= \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right] \end{aligned} \tag{7.3.22}$$

We can rearrange this equation to the form

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2x^2\right)\psi = 0 \tag{7.3.3}$$

We wish to find all solutions to this equation that lie in the physical Hilbert space (of functions normalizable to unity or the Dirac delta function). Follow the approach closely—it will be invoked often in the future.

The first step is to write Eq. (7.3.3) in terms of dimensionless variables. We look for a new variable y which is dimensionless and related to x by

$$x = by \tag{7.3.4}$$

where b is a scale factor with units of length. Although any length b (say the radius of the solar system) will generate a dimensionless variable y, the idea is to choose the natural length scale generated by the equation itself. By feeding Eq. (7.3.4) into Eq. (7.3.3), we arrive at

$$\frac{d^2\psi}{dy^2} + \frac{2mEb^2}{\hbar^2}\psi - \frac{m^2\omega^2b^4}{\hbar^2}y^2\psi = 0 \tag{7.3.5}$$

The last terms suggests that we choose

$$b = \left(\frac{\hbar}{m\omega}\right)^{1/2} \tag{7.3.6}$$

Let us also define a dimensional variable ε corresponding to E:

$$\varepsilon = \frac{mEb^2}{\hbar^2} = \frac{E}{\hbar\omega} \tag{7.3.7}$$

(We may equally well choose $\varepsilon = 2mEb^2/\hbar^2$. Constants of order unity are not uniquely suggested by the equation. In the present case, our choice of ε is in anticipation of the results.) In terms of the dimensionless variables, Eq. (7.3.5) becomes

$$\psi'' + (2\varepsilon - y^2)\psi = 0 \tag{7.3.8}$$

where the prime denotes differentiation with respect to y.

Not only do dimensionless variables lead to a more compact equation, they also provide the natural scales for the problem. By measuring x and E in units of $(\hbar/m\omega)^{1/2}$ and $\hbar\omega$, which are scales generated intrinsically by the parameters entering the problem, we develop a feeling for what the words "small" and "large" mean: for example the displacement of the oscillator

where u approaches $A + cy$ (plus higher powers) as $y \to 0$, and y^m (plus lower powers) as $y \to \infty$. To determine $u(y)$ completely, we feed the above *ansatz* into Eq. (7.3.8) and obtain

$$u'' - 2yu' + (2\varepsilon - 1)u = 0 \tag{7.3.11}$$

This equation has the desired features (to be discussed in Exercise 7.3.1) that indicate that a power-series solution is possible, i.e., if we assume

$$u(y) = \sum_{n=0}^{\infty} C_n y^n \tag{7.3.12}$$

the equation will determine the coefficients. [The series begins with $n = 0$, and not some negative n, since we know that as $y \to 0$, $u \to A + cy + O(y^2)$.] Feeding this series into Eq. (7.3.11) we find

$$\sum_{n=0}^{\infty} C_n[n(n-1)y^{n-2} - 2ny^n + (2\varepsilon - 1)y^n] = 0 \tag{7.3.13}$$

Consider the first of three pieces in the above series:

$$\sum_{n=0}^{\infty} C_n n(n-1)y^{n-2}$$

Due to the $n(n-1)$ factor, this series also equals

$$\sum_{n=2}^{\infty} C_n n(n-1)y^{n-2}$$

In terms of a new variable $m = n - 2$ the series becomes

$$\sum_{m=0}^{\infty} C_{m+2}(m+2)(m+1)y^m \equiv \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)y^n$$

since m is a dummy variable. Feeding this equivalent series back into Eq. (7.3.13) we get

$$\sum_{n=0}^{\infty} y^n[C_{n+2}(n+2)(n+1) + C_n(2\varepsilon - 1 - 2n)] = 0 \tag{7.3.14}$$

Since the functions y^n are linearly independent (you cannot express y^n as a linear combination of other powers of y) each coefficient in the linear relation above must vanish. We thus find

$$C_{n+2} = C_n \frac{(2n + 1 - 2\varepsilon)}{(n+2)(n+1)} \tag{7.3.15}$$

The derivation of the normalization constant

$$A_n = \left[\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right]^{1/4} \tag{7.3.23}$$

is rather tedious and will not be discussed here in view of a shortcut to be discussed in the next section.

The following recursion relations among Hermite polynomials are very useful:

$$H_n'(y) = 2nH_{n-1} \tag{7.3.24}$$

$$H_{n+1}(y) = 2yH_n - 2nH_{n-1} \tag{7.3.25}$$

as is the integral

$$\int_{-\infty}^{\infty} H_n(y)H_{n'}(y)e^{-y^2}\,dy = \delta_{nn'}(\pi^{1/2}2^n n!) \tag{7.3.26}$$

which is just the orthonormality condition of the eigenfunctions $\psi_n(x)$ and $\psi_{n'}(x)$ written in terms of $y = (m\omega/\hbar)^{1/2}x$.

We can now express the propagator as

$$U(x, t; x', t') = \sum_{n=0}^{\infty} A_n \exp\left(-\frac{m\omega}{2\hbar}x^2\right)H_n(x)A_n \exp\left(-\frac{m\omega}{2\hbar}x'^2\right) \times H_n(x')\exp[-i(n+1/2)\omega(t-t')] \tag{7.3.27}$$

Evaluation of this sum is a highly formidable task. We will not attempt it here since we will find an extremely simple way for calculating U in Chapter 8, devoted to the path integral formalism. The result happens to be

$$U(x, t; x', t') = \left(\frac{m\omega}{2\pi i\hbar \sin \omega T}\right)^{1/2} \exp\left[\frac{im\omega}{\hbar}\,\frac{(x^2 + x'^2)\cos \omega T - 2xx'}{2 \sin \omega T}\right]$$

where $T = t - t'$. (7.3.28)

This concludes the solution of the eigenvalue problem. Before analyzing our results let us recapitulate our strategy.

Step 1: Introduce dimensionless variables natural to the problem.

Step 2: Extract the asymptotic ($y \to \infty$, $y \to 0$) behavior of ψ.

Step 3: Write ψ as a product of the asymptotic form and an unknown function u. The function u will usually be easier to find than ψ.

Step 4: Try a power series to see if it will yield a recursion relation of the form Eq. (7.3.15).