Chapter 2 Relativity II

In the opening section of Chapter 1 we discussed the classical observation that, if Newton's second law $\mathbf{F} = m\mathbf{a}$ holds in a particular reference frame, it also holds in any other reference frame that moves with constant velocity relative to it, i.e., in any inertial frame. As shown in Section 1-1, the Galilean transformation (Equations 1-3) leads to the same accelerations $a'_x = a_x$ in both frames, and forces such as those due to stretched springs are also the same in both frames. However, according to the Lorentz transformation, accelerations are not the same in two such reference frames. If a particle has acceleration a_x and velocity u_x in frame S, its acceleration in S', obtained by computing du'_x/dt' from Equation 1-24, is

$$a'_x = \frac{a_x}{\gamma^3(1 - vu_x/c^2)^3} \quad \text{2-1}$$

Thus, F/m must transform in a similar way, or Newton's second law $\mathbf{F} = m\mathbf{a}$ does not hold.

It is reasonable to expect that $\mathbf{F} = m\mathbf{a}$ does *not* hold at high speeds, for this equation implies that a constant force will accelerate a particle to unlimited velocity if it acts for a long time. However, if a particle's velocity was greater than c in some reference frame S, we could not transform from S to the rest frame of the particle because γ becomes imaginary when $v > c$. We can show from the velocity transformation that if a particle's velocity is less than c in some frame S, it is less than c in all frames moving relative to S with $v < c$. This result leads us to expect that particles never have speeds greater than c. Thus, we expect that Newton's second law $\mathbf{F} = m\mathbf{a}$ is not relativistically invariant. We will, therefore, need a new law of motion, but one that reduces to Newton's classical version when $\beta\ (=v/c) \rightarrow 0$, since $\mathbf{F} = m\mathbf{a}$ is consistent with experimental observations when $\beta \ll 1$.

In this chapter we will explore the changes in classical dynamics that are dictated by relativity theory, directing particular attention to the same concepts around which classical mechanics was developed, namely mass, momentum, and energy. We will find these changes to be every bit as dramatic as those we encountered in Chapter 1, including a Lorentz transformation for momentum and energy and a new invariant quantity to stand beside the invariant spacetime interval Δs. Then, in the latter part of the chapter, we will briefly turn our attention to noninertial, or accelerated, reference frames, the realm of the theory of general relativity.

2-1 Relativistic Momentum

Among the most powerful fundamental concepts that you have studied in physics until now have been the ideas of conservation of momentum and conservation of total energy. As we will discuss a bit further in Chapter 13, each of these fundamental laws arises because of a particular symmetry that exists in the laws of physics. For example, the conservation of total energy in classical physics is a consequence of the symmetry, or invariance, of the laws of physics to translations in time. As a consequence, Newton's laws work exactly the same way today as they did when he first wrote them down. The conservation of momentum arises from the invariance of physical laws to translations in space. Indeed, Einstein's first postulate and the resulting Lorentz transformation (Equations 1-20 and 1-21) guarantee this latter invariance in all inertial frames.

The simplicity and universality of these conservation laws leads us to seek equations for relativistic mechanics, replacing Equation 1-1 and others, that are consistent with momentum and energy conservation and are also invariant under a Lorentz transformation. However, it is straightforward to show that the momentum, as formulated in classical mechanics, does not result in relativistic invariance of the law of conservation of momentum. To see that this is so, we will look at an isolated collision between two masses, where we avoid the question of how to transform forces because the net external force is zero. In classical mechanics, the total momentum $\mathbf{p} = \Sigma m_i \mathbf{u}_i$ is conserved. We can see that relativistically, conservation of the quantity $\Sigma m_i \mathbf{u}_i$ is an approximation which holds only at low speeds.

Consider one observer in frame S with a ball A and another in S' with a ball B. The balls each have mass m and are identical when measured at rest. Each observer throws his ball along his y axis with speed u_0 (measured in his own frame) so that the balls collide.[1] Assuming the balls to be perfectly elastic, each observer will see his ball rebound with its original speed u_0. If the total momentum is to be conserved, the y component must be zero because the momentum of each ball is merely reversed by the collision. However, if we consider the relativistic velocity transformations, we can see that the quantity mu_y does not have the same magnitude for each ball as seen by either observer.

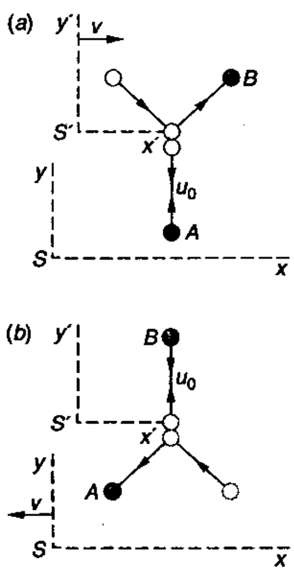


Fig. 2-1 (a) Elastic collision of two identical balls as seen in frame S. The vertical component of the velocity of ball B is u_0/γ in S if it is u_0 in S'. (b) The same collision as seen in S'. In this frame ball A has vertical component of velocity u_0/γ.

Let us consider the collision as seen in frame S (Figure 2-1a). In this frame ball A moves along the y axis with velocity $u_{yA} = u_0$. Ball B has x component of velocity $u_{xB} = v$ and y component

$$u_{yB} = u'_{yB}/\gamma = -u_0\sqrt{1 - v^2/c^2} \tag{2-2}$$

Here we have used the velocity transformation equations (1-25) and the facts the u'_{yB} is just $-u_0$ and $u'_{xB} = 0$. We see that the y component of velocity of ball B is smaller in magnitude than that of ball A. The quantity $(1 - v^2/c^2)^{1/2}$ comes from the time dilation factor. The time taken for ball B to travel a given distance along the y axis in S is greater than the time measured in S' for the ball to travel this same distance. Thus in S the total y component of classical momentum is not zero. Since the y components of the velocities are reversed in an elastic collision, momentum as defined by $\mathbf{p} = \Sigma m\mathbf{u}$ is not conserved in S. Analysis of this problem in S' leads to the same conclusion (Figure 2-1b), since the roles of A and B are simply interchanged.[2] In the classical limit $v \ll c$, momentum is conserved, of course, because in that limit $\gamma \approx 1$ and $u_{yB} \approx u_0$.

The reason for defining momentum as $\Sigma m\mathbf{u}$ in classical mechanics is that this quantity is conserved when there is no net external force, as in our collision example.

We now see that this quantity is conserved only in the approximation $v \ll c$. We shall define *relativistic momentum* **p** of a particle to have the following properties:

1. **p** is conserved in collisions.
2. **p** approaches $m\mathbf{u}$ as u/c approaches zero.

Let's apply the first of these conditions to the collision of the two balls that we just discussed, noting two important points. First, for each observer in Figure 2-1, the speed of each ball is unchanged by the elastic collision. It is either u_0 (for the observer's own ball) or $(u_y^2 + v^2)^{1/2} = u$ (for the other ball). Second, the failure of the conservation of momentum in the collision we described can't be due to the velocities because we used the Lorentz transformation to find the y components. It must have something to do with the mass! Let us write down the conservation of the y component of the momentum *as observed in S*, keeping the masses of the two balls straight by writing $m(u_0)$ for the S observer's own ball and $m(u)$ for the S' observer's ball.

$$m(u_0)u_0 - m(u)u_{yB} = -m(u_0)u_0 + m(u)u_{yB} \qquad \text{2-3}$$

(before collision) **(after collision)**

Equation 2-3 can be readily rewritten as

$$\frac{m(u)}{m(u_0)} = \frac{u_0}{u_{yB}} \qquad \text{2-4}$$

If u_0 is small compared to the relative speed v of the reference frames, then it follows from Equation 2-2 that $u_{yB} \ll v$ and, therefore, $u \approx v$.

If we can now imagine the limiting case where $u_0 \rightarrow 0$, i.e., where each ball is at rest in its "home" frame so that the collision becomes a "grazing" one as B moves past A at speed $v = u$, then we conclude from Equations 2-2 and 2-4 that in order for Equation 2-3 to hold, i.e., for the momentum to be conserved,

$$\frac{m(u = v)}{m(u_0 = 0)} = \frac{u_0}{u_0\sqrt{1 - v^2/c^2}}$$

or

$$m(u) = \frac{m}{\sqrt{1 - u^2/c^2}} \quad = \gamma m \qquad \text{2-5}$$

Equation 2-5 says that the observer in S *measures* the mass of ball B, moving relative to him at speed u, as equal to $1/(1 - u^2/c^2)^{1/2}$ times the rest mass of the ball, or its mass measured in the frame in which it is at rest. Notice that observers always measure the mass of an object that is in motion with respect to them to be larger than the value measured when the object is at rest.

Thus we see that the law of conservation of momentum will be valid in relativity, provided that we write the momentum **p** of an object with rest mass m moving with velocity **u** relative to an inertial system S to be

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} \quad = \gamma m u \qquad \text{2-6}$$

The design and construction of the large particle accelerators throughout the world are based directly on the relativistic expressions for momentum and energy.

where u is the speed of the particle. We thus take this equation for the definition of relativistic momentum. It is clear that this definition meets our second criterion, because the denominator approaches 1 when u is much less than c. From this definition, the momenta of the two balls A and B in Figure 2-1 as seen in S are

$$p_{yA} = \frac{mu_0}{\sqrt{1 - u_0^2/c^2}} \qquad p_{yB} = \frac{mu_{yB}}{\sqrt{1 - (u_{xB}^2 + u_{yB}^2)/c^2}}$$

where $u_{yB} = u_0(1 - v^2/c^2)^{1/2}$ and $u_{xB} = v$. It is similarly straightforward to show that $p_{yB} = -p_{yA}$. Because of the similarity of the factor $1/(1 - u^2/c^2)^{1/2}$ and γ in the Lorentz transformation, Equation 2-6 is often written

$$\mathbf{p} = \gamma m\mathbf{u} \qquad \text{with} \qquad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{2-7}$$

This use of the symbol γ for two different quantities causes some confusion; the notation is standard, however, and simplifies many of the equations. We shall use this notation except when we are also considering transformations between reference frames. Then, to avoid confusion, we shall write out the factor $(1 - u^2/c^2)^{1/2}$ and reserve γ for $1/(1 - v^2/c^2)^{1/2}$, where v is the relative speed of the frames. Figure 2-2 shows a graph of the magnitude of **p** as a function of u/c. The quantity $m(u)$ in Equation 2-5 is sometimes called the *relativistic mass;* however, we shall avoid using the term or a symbol for relativistic mass: in this book, *m always* refers to the mass measured in the rest frame. In this we are following Einstein's view. In a letter to a colleague in 1948 he wrote:[3]

> It is not good to introduce the concept of mass $M = m/(1 - v^2/c^2)^{1/2}$ of a body for which no clear definition can be given. It is better to introduce no other mass than "the rest mass" m. Instead of introducing M, it is better to mention the expression for the momentum and energy of a body in motion.

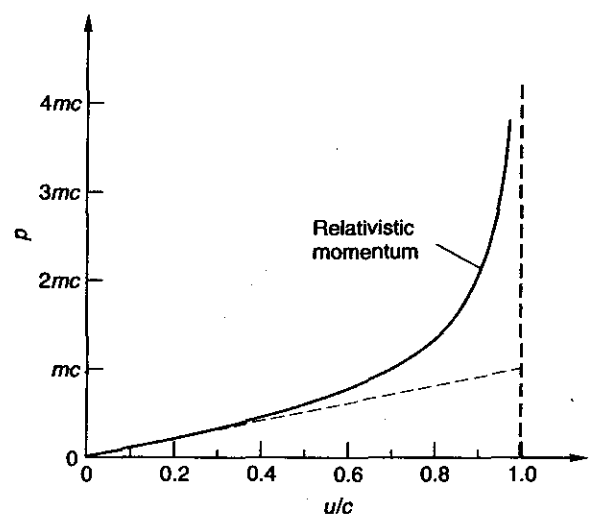


Fig. 2-2 Relativistic momentum as given by Equation 2-6 versus u/c, where u = speed of the object relative to an observer. The magnitude of the momentum p is plotted in units of mc. The fainter dashed line shows the classical momentum mu for comparison.

EXAMPLE 2-1 Measured Values of Moving Mass For what value of u/c will the measured mass of an object γm exceed the rest mass by a given fraction f?

Solution

From Equation 2-5 we see that

$$f = \frac{\gamma m - m}{m} = \gamma - 1 = \frac{1}{\sqrt{1 - u^2/c^2}} - 1$$

Solving for u/c,

$$1 - u^2/c^2 = \frac{1}{(f + 1)^2} \longrightarrow u^2/c^2 = 1 - \frac{1}{(f + 1)^2}$$

or

$$u/c = \frac{\sqrt{f(f + 2)}}{f + 1}$$

from which we can compute the table of values below or the value of u/c for any other f. Note that the value of u/c that results in a given fractional increase f in the measured value of the mass is independent of m. A diesel locomotive moving at a particular u/c will be observed to have the same f as a proton moving with that u/c.

[illegible]	[illegible]	[illegible]
10^{-12}	1.4×10^{-6}	jet fighter aircraft
5×10^{-9}	0.0001	Earth's orbital speed
0.0001	0.014	50-eV electron
0.01 (1%)	0.14	quasar 3C 273
1.0 (100%)	0.87	quasar OQ172
10.0	0.996	muons from cosmic rays
100	0.99995	some cosmic ray protons

EXAMPLE 2-2 Momentum of a Rocket A high-speed interplanetary probe with a mass m = 50,000 kg has been sent toward Pluto at a speed $u = 0.8c$. What is its momentum as measured by Mission Control on Earth? If, preparatory to landing on Pluto, the probe's speed is reduced to $0.4c$, by how much does its momentum change?

Solution

1. Assuming that the probe travels in a straight line toward Pluto, its momentum along that direction is given by Equation 2-6:

$$p = \frac{mu}{\sqrt{1 - u^2/c^2}}$$
$$= \frac{(50{,}000 \text{ kg})(0.8c)}{\sqrt{1 - (0.8)^2}}$$
$$= 6.7 \times 10^4\, c \cdot \text{kg}$$
$$= 2.0 \times 10^{13}\, \text{kg} \cdot \text{m/s}$$

2. When the probe's speed is reduced, the momentum declines a ong the relativistic momentum curve in Figure 2-2. The new value is compu ed from the ratio:

$$\frac{p_{0.4c}}{p_{0.8c}} = \frac{m(0.4c)/\sqrt{1 - (0.4)^2}}{m(0.8c)/\sqrt{1 - (0.8)^2}}$$
$$= \frac{1}{2}\frac{\sqrt{1 - (0.8)^2}}{\sqrt{1 - (0.4)^2}}$$
$$= 0.33$$

3. The reduced momentum $p_{0.4c}$ is then given by:

$$p_{0.4c} = 0.33\, p_{0.8c}$$
$$= (0.33)(6.7 \times 10^4\, c \cdot \text{kg})$$
$$= 2.2 \times 10^4\, c \cdot \text{kg}$$
$$= 6.6 \times 10^{12}\, \text{kg} \cdot \text{m/s}$$

Remarks: *Notice from Figure 2-2 that the incorrect classical value of $p_{0.4c}$ would have been 4.0×10^4 c · kg. Also, while the probe's speed was decreased to one-half its initial value, the momentum decreased to one-third of the initial va.ue.*

1. In our discussion of the inelastic collision of balls A and B, the collision was a "grazing" one in the limiting case. Suppose instead that the collisi n is a "head-on" one along the x axis. If the speed of S' (i.e., ball B) is low, say, $v = 0.1c$, what would a spacetime diagram of the collision look like?

2-2 Relativistic Energy

As noted in the preceding section, the fundamental character of the principle of conservation of total energy leads us to seek a definition of total energy in relativ ity that preserves the invariance of that conservation law in transformations between inertial systems. As with the definition of the relativistic momentum, Equation 2-6, we shall require that the *relativistic total energy E* satisfy two conditions:

1. The total energy E of any isolated system is conserved.
2. E will approach the classical value when u/c approaches zero.

Let's first find a form for E that satisfies the second condition and then see if it also satisfies the first. We have seen that the quantity $m\mathbf{u}$ is not conserved in collisions but that $\gamma m\mathbf{u}$ is, with $\gamma = 1/(1 - u^2/c^2)^{1/2}$. We have also noted that Newton's second law in the form $\mathbf{F} = m\mathbf{a}$ cannot be correct relativistically, one reason being that it leads to the conservation of $m\mathbf{u}$. We can get a hint of the relativistically correct form of the second law by writing it $\mathbf{F} = d\mathbf{p}/dt$. This equation is relativistically correct *if* relativistic momentum $\mathbf{p}$ is used. We thus define the force in relativity to be

$$F = \frac{d\mathbf{p}}{dt} = \frac{d(\gamma m\mathbf{u})}{dt} \qquad \textbf{2-8}$$

Now then, as in classical mechanics, we shall define kinetic energy E_k as the work done by a net force in accelerating a particle from rest to some velocity u. Considering motion in one dimension only, we have

$$E_k = \int_{u=0}^{u} F\,dx = \int_0^u \frac{d(\gamma mu)}{dt}\,dx = \int_0^u u\,d(\gamma mu)$$

Aerial view of the Stanford Linear Accelerator Center (SLAC). Electrons begin their 3-km acceleration to relativistic energies in the upper right, cross under Interstate 280 at nearly the speed of light, and fan out to several experiment stations in the foreground, where their collisions in the underground storage ring create short-lived mesons. [*Stanford Linear Accelerator Center, U.S. Department of Energy.*]

using $u = dx/dt$. The computation of the integral in this equation is not difficult but requires a bit of algebra. It is left as an exercise (Problem 2-2) to show that

$$d(\gamma mu) = m\left(1 - \frac{u^2}{c^2}\right)^{-3/2} du$$

Substituting this into the integrand in Equation 2-8, we obtain

$$E_k = \int_0^u u\, d(\gamma mu) = \int_0^u m\left(1 - \frac{u^2}{c^2}\right)^{-3/2} u\, du$$
$$= mc^2\left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1\right)$$

or

$$E_k = \gamma mc^2 - mc^2 \qquad \textbf{2-9}$$

Equation 2-9 defines the ***relativistic kinetic energy.*** Notice that, as we warned earlier, E_k is ***not*** $mu^2/2$ or even $\gamma mu^2/2$. This is strikingly evident in Figure 2-3. However, consistent with our second condition on the relativistic total energy E, Equation 2-9 does approach $mu^2/2$ when $u \ll c$. We can check this assertion by noting that for $u/c \ll 1$, expanding γ by the binomial theorem yields

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

and thus

$$E_k = mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots - 1\right) \approx \frac{1}{2}mu^2$$

The expression for kinetic energy in Equation 2-9 consists of two terms. One term, γmc^2, depends on the speed of the particle (through the factor γ), and the other term, mc^2, is independent of the speed. The quantity mc^2 is called the ***rest energy*** of the particle, i.e., the energy associated with the rest mass m. The relativistic total energy E is then defined as the sum of the kinetic energy and the rest energy:

$$E = E_k + mc^2 = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} \qquad \textbf{2-10}$$

Thus, the work done by a net force increases the energy of the system from the rest energy mc^2 to γmc^2 (or increases the measured mass from m to γm).

For a particle at rest relative to an observer, $E_k = 0$, and Equation 2-10 becomes perhaps the most widely recognized equation in all of physics, Einstein's famous $E = mc^2$. When $u \ll c$ Equation 2-10 can be written as

$$E \approx \frac{1}{2}mu^2 + mc^2$$

Before the development of relativity theory, it was thought that mass was a conserved quantity;[4] consequently, m would always be the same before and after an interaction

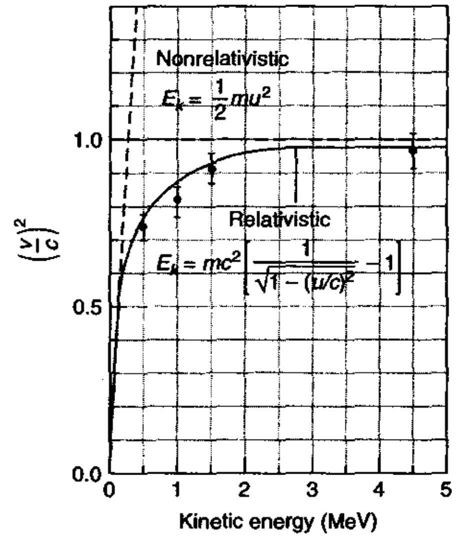


Fig. 2-3 Experimental confirmation of the relativistic relation for kinetic energy. Electrons were accelerated to energies up to several MeV in large electric fields and their velocities were determined by measuring their time of flight over 8.4 m. Note that when the velocity $u \ll c$, the relativistic and nonrelativistic (i.e., classical) relations are indistinguishable. [*W. Bertozzi, American Journal of Physics, 32, 551 (1964).*]

or event and mc^2 would therefore be constant. Since the zero of energy is arbitrary, we are always free to include an additive constant; therefore, our definition of the relativistic total energy reduces to the classical kinetic energy for $u \ll c$ and our second condition on E is thus satisfied.[5]

Be very careful to understand Equation 2-10 correctly. It defines the total energy E, and E is what we are seeking to conserve for isolated systems in all inertial frames, *not* E_k and *not* mc^2. Remember, too, the distinction between *conserved* quantities and *invariant* quantities. The former have the same value before and after an interaction in a particular reference frame. The latter have the same value when measured by observers in different reference frames. Thus, we are not requiring observers in relatively moving inertial frames to measure the same values for E, but rather that E be unchanged in interactions as measured in each frame. To assist us in showing that E as defined by Equation 2-10 is conserved in relativity, we will first see how E and **p** transform between inertial reference frames.

Lorentz Transformation of E and **p**

Consider a particle of rest mass m that has an arbitrary velocity **u** with respect to frame S as shown in Figure 2-4. System S' is a second inertial frame moving in the $+x$ direction. The particle's momentum and energy are given in the S and S' systems, respectively, by:

In S:

$$\begin{aligned} E &= \gamma mc^2 \\ p_x &= \gamma m u_x \\ p_y &= \gamma m u_y \\ p_z &= \gamma m u_z \end{aligned} \qquad \textbf{2-11}$$

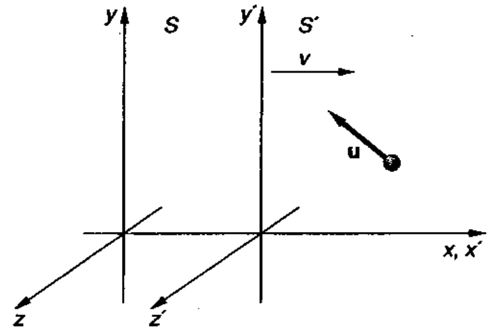


Fig. 2-4 Particle of mass m moves with velocity **u** measured in S. System S' moves in the $+x$ direction at speed v. The Lorentz velocity transformation enables determination of the relations connecting measurements of the total energy and the components of the momentum in the two frames of reference.

where

$$\gamma = 1/\sqrt{1 - u^2/c^2}$$

In S':

$$\begin{aligned} E' &= \gamma' mc^2 \\ p'_x &= \gamma' mu'_x \\ p'_y &= \gamma' mu'_y \\ p'_z &= \gamma' mu'_z \end{aligned} \qquad \textbf{2-12}$$

where

$$\gamma' = 1/\sqrt{1 - u'^2/c^2}$$

Developing the Lorentz transformations for E and **p** requires that we first express γ' in terms of quantities measured in S. (We could just as well express γ in terms of primed quantities. Since this is relativity, it makes no difference which we choose.) The result is

$$\frac{1}{\sqrt{1 - u'^2/c^2}} = \gamma \frac{(1 - vu_x/c^2)}{\sqrt{1 - u^2/c^2}} \quad \text{where now} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \textbf{2-13}$$

Substituting Equation 2-13 into the expression for E' in Equations 2-12 yields

$$E' = \frac{mc^2}{\sqrt{1 - u'^2/c^2}} = \gamma\left[\frac{mc^2}{\sqrt{1 - u^2/c^2}} - \frac{mc^2 vu_x/c^2}{\sqrt{1 - u^2/c^2}}\right]$$

The first term in the brackets you will recognize as E, and the second term, canceling the c^2 factors, as vp_x from Equations 2-11. Thus, we have

$$E' = \gamma(E - vp_x) \qquad \textbf{2-14}$$

Similarly, substituting Equation 2-13 and the velocity transformation for u'_x into the expression for p'_x in Equations 2-12 yields

$$p'_x = \frac{mu'_x}{\sqrt{1 - u'^2/c^2}} = \gamma\left[\frac{mu_x}{\sqrt{1 - u^2/c^2}} - \frac{mv}{\sqrt{1 - u^2/c^2}}\right]$$

The first term in the brackets is p_x from Equations 2-11 and, noting that $m(1 - u^2/c^2)^{-1/2} = E/c^2$, the second term is vE/c^2. Thus we have

$$p'_x = \gamma(p_x - vE/c^2) \qquad \textbf{2-15}$$

Using the same approach, it can be shown (Problem 2-45) that

$$p'_y = p_y \qquad \text{and} \qquad p'_z = p_z$$

Together these relations are the *Lorentz transformation for momentum and energy:*

$$\begin{aligned} p'_x &= \gamma(p_x - vE/c^2) \qquad & p'_y &= p_y \\ E' &= \gamma(E - vp_x) \qquad & p'_z &= p_z \end{aligned} \qquad \textbf{2-16}$$

The inverse transformation is

$$\begin{aligned} p_x &= \gamma(p'_x + vE'/c^2) \qquad & p_y &= p'_y \\ E &= \gamma(E' + vp'_x) \qquad & p_z &= p'_z \end{aligned}$$

with

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}} \qquad \textbf{2-17}$$

Note the striking similarity between Equations 2-16 and 2-17 and the Lorentz transformation of the space and time coordinates, Equations 1-20 and 1-21. The momentum $\mathbf{p}(p_x, p_y, p_z)$ transforms in relativity exactly like $\mathbf{r}(x, y, z)$, and the total energy E transforms like the time t. We will return to this remarkable result and related matters shortly, but first let's do some examples and then, as promised, show that the energy defined by Equation 2-10 is conserved in relativity.

EXAMPLE 2-3 **Transforming Energy and Momentum** Suppose a micrometeorite of mass 10^{-9} kg moves past Earth at a speed of $0.01c$. What values will be measured for the energy and momentum of the particle by an observer in a system S' moving relative to Earth at $0.5c$ in the same direction as the micrometeorite?

Solution

Taking the direction of the micrometeorite's travel as the x axis, its energy and momentum as measured by the Earth observer are, using the $u \ll c$ approximation of Equation 2-10:

$$E \approx \tfrac{1}{2}mu^2 + mc^2 = 10^{-9}\,\text{kg}[(0.01c)^2/2 + c^2]$$
$$E \approx 1.00005 \times 10^{-9}\,c^2\ \text{J}$$

and

$$p_x = mu_x = (10^{-9}\ \text{kg})(0.01c) = 10^{-11}\ c\ \text{kg}\cdot\text{m/s}$$

For this situation $\gamma = 1.1547$, so in S' the measured values of the energy and momentum will be:

$$E' = \gamma(E - vp_x) = (1.1547)[1.00005 \times 10^{-9}c^2 - (0.5c)(10^{-11}c)]$$
$$E' = (1.1547)(1.00005 \times 10^{-9} - 0.5 \times 10^{-11})c^2$$
$$E' = 1.14898 \times 10^{-9}\,c^2\ \text{J}$$

and

$$p'_x = \gamma(p_x - vE/c^2) = (1.1547)[10^{-11}c - (0.05c)(1.00005 \times 10^{-9}\ c^2)/c^2]$$
$$p'_x = (1.1547)(10^{-11} - 5.00025 \times 10^{-10})c$$
$$p'_x = -0.566 \times 10^{-11}\,c\ \text{kg}\cdot\text{m/s}$$

Thus, the observer in S' measures a total energy nearly 15 percent larger and a momentum about 40 percent smaller and in the $-x$ direction.

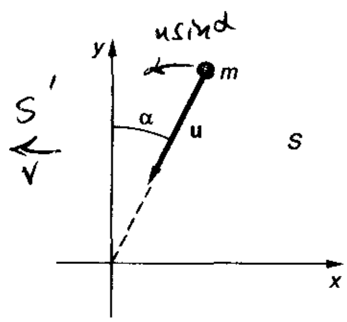


Fig. 2-5 The system discussed in Example 2-4.

EXAMPLE 2-4 A More Difficult Lorentz Transformation of Energy Suppose that a particle with mass m and energy E is moving toward the origin of a system S such that its velocity **u** makes an angle α with the y axis as shown in Figure 2-5. Using the Lorentz transformation for energy and momentum, determine the energy E' of the particle measured by an observer in S', which moves relative to S so that the particle moves along the y' axis.

Solution

System S' moves in the $-x$ direction at speed $u \sin \alpha$, as determined from the Lorentz velocity transformation for $u'_x = 0$. Thus, $v = -u \sin \alpha$. Also,

$$E = mc^2/\sqrt{1 - u^2/c^2} \qquad p = mu/\sqrt{1 - u^2/c^2}$$

and from the latter,

$$p_x = -\left(mu/\sqrt{1 - u^2/c^2}\right)\sin\alpha$$

In S' the energy will be

$$E' = \gamma(E - vp_x)$$
$$= \frac{1}{\sqrt{1 - v^2/c^2}}[E - (-u\sin\alpha)(-mu/\sqrt{1 - u^2/c^2})\sin\alpha]$$
$$= \frac{1}{\sqrt{1 - u^2\sin^2\alpha/c^2}}[E - (m/\sqrt{1 - u^2/c^2})u^2\sin^2\alpha]$$

Multiplying the second term in the brackets by c^2/c^2 and factoring an E from both terms yield

$$E' = E\sqrt{1 - (u^2/c^2)\sin^2 \alpha}$$

Since $u < c$ and $\sin^2 \alpha \leq 1$, we see that $E' < E$, except for $\alpha = 0$ when $E' = E$, in which case S and S' are the same system. Note, too, that for $\alpha > 0$, if $u \to c$, $E' \to E \cos \alpha$. As we will see later, this is the case for light.

QUESTION

2. Recalling the results of the measurements of time and space intervals by observers in motion relative to clocks and measuring rods, discuss the results of corresponding measurements of energy and momentum changes.

Conservation of Energy

As with our discussion of momentum conservation in relativity, let us consider a collision of two identical particles, each with rest mass m. This time, for a little variety, we will let the collision be completely inelastic—i.e., when the particles collide, they stick together. There is a system S', called the *zero-momentum frame,* in which the particles approach each other along the x' axis with equal speeds u—hence equal and opposite momenta—as illustrated in Figure 2-6*a*. In this frame the collision results in the formation of a composite particle of mass M at rest in S'. If S' moves with respect to a second frame S at speed $v = u$ in the x direction, then the particle on the right before the collision will be at rest in S and the composite particle will move to the right at speed u in that frame. This situation is illustrated in Figure 2-6*b*.

KE not conserved

Using the total energy as defined by Equation 2-10, we have in S'.

Before collision:

$$E'_{\text{before}} = \frac{mc^2}{\sqrt{1 - u^2/c^2}} + \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \frac{2mc^2}{\sqrt{1 - u^2/c^2}} \tag{2-18}$$

After collision:

$$E'_{\text{after}} = Mc^2 \tag{2-19}$$

Energy will be conserved in S' if $E'_{\text{before}} = E'_{\text{after}}$, i.e., if

$$\frac{2mc^2}{\sqrt{1 - u^2/c^2}} = Mc^2 \tag{2-20}$$

This is ensured by the validity of conservation of momentum, in particular by Equation 2-5, and so energy is conserved in S'. (The validity of Equation 2-20 is important and not trivial. We will consider it in more detail in Example 2-7.) To see if energy as

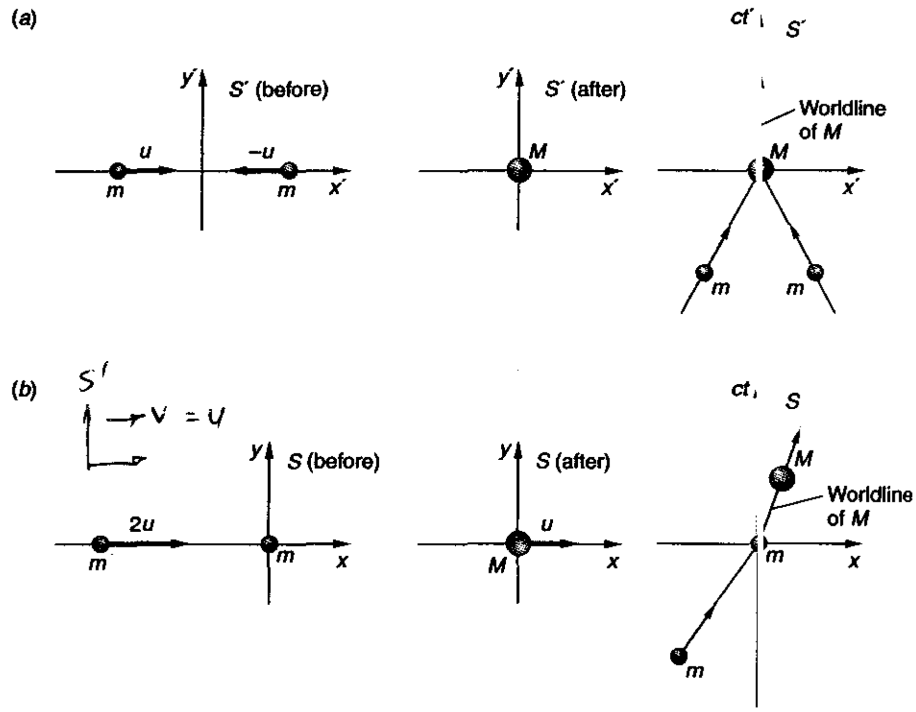


Fig. 2-6 Inelastic collision of two particles of equal rest mass m. (a) In the zero momentum frame S' the particles have equal and opposite velocities and, hence, momenta. After the collision, the composite particle of mass M is at rest in S'. The diagram on the far right is the spacetime diagram of the collision from the viewpoint of S'. (b) In system S the frame S' is moving to the right at speed u so that the particle on the right is at rest in S, while the left one moves at $2u$. After collision, the composite particle moves to the right at speed u. Again, the spacetime diagram of the interaction is shown on the far right. All diagrams are drawn with the collision occurring at the origin.

we have defined it is also conserved in S, we transform to S using the inverse energy transform, Equation 2-17. We then have in S:

Before collision:

$$E_{\text{before}} = \gamma(E'_{\text{before}} + vp'_x)$$

$$E_{\text{before}} = \gamma\left(\frac{2mc^2}{\sqrt{1-u^2/c^2}} + up'_x\right)$$

$$= \gamma\left(\frac{2mc^2}{\sqrt{1-u^2/c^2}}\right) \quad \text{since} \quad p'_x = 0 \qquad \textbf{2-21}$$

After collision:

$$E_{\text{after}} = \gamma(Mc^2 + up'_x) = \gamma Mc^2 \quad \text{since again} \quad p'_x = 0 \qquad \textbf{2-22}$$

The energy will be conserved in S and, therefore, the law of conservation of energy will hold in all inertial frames if $E_{\text{before}} = E_{\text{after}}$, i.e., if

$$\gamma\left(\frac{2mc^2}{\sqrt{1-u^2/c^2}}\right) = \gamma Mc^2 \qquad \textbf{2-23}$$

which, like Equation 2-20, is ensured by Equation 2-5. Thus, we conclude that the energy as defined by Equation 2-10 is consistent with a relativistically invariant law of conservation of energy, satisfying the first of the conditions set forth at the beginning of this section. While this demonstration has not been a general one, that being beyond the scope of our discussions, you may be assured that our conclusion is quite generally valid.

Question

1. Explain why the result of Example 2-4 does not mean that energy conservation is violated.

EXAMPLE 2-5 **Mass of Cosmic Ray Muons** In Chapter 1, muons produced as secondary particles by cosmic rays were used to illustrate both the relativistic length contraction and time dilation resulting from their high speed relative to observers on Earth. That speed is about $0.998c$. If the rest energy of a muon is 105.7 MeV, what will observers on Earth measure for the total energy of a cosmic ray–produced muon? What will they measure for its mass?

Solution

The electron volt (eV), the amount of energy acquired by a particle with electric charge equal in magnitude to that on an electron (e) accelerated through a potential difference of 1 volt, is a convenient unit in physics, as you may have learned. It is defined as

$$1.0\text{ eV} = 1.602 \times 10^{-19}\text{ C} \times 1.0\text{ V} = 1.602 \times 10^{-19}\text{ J} \qquad \textbf{2-24}$$

Commonly used multiples of the eV are the keV (10^3 eV), the MeV (10^6 eV), the GeV (10^9 eV), and the TeV (10^{12} eV). Many experiments in physics involve the measurement and analysis of the energy and/or momentum of particles and systems of particles, and Equation 2-10 allows us to express the masses of particles in energy units, rather than the SI unit of mass, the kilogram. That and the convenient size of the eV facilitate[6] numerous calculations. For example, the mass of an electron is 9.11×10^{-31} kg. Its rest energy is given by

$$E = mc^2 = 9.11 \times 10^{-31}\text{ kg}\cdot c^2 = 8.19 \times 10^{-14}\text{ J}$$

or

$$E = 8.19 \times 10^{-14}\text{ J} \times \frac{1}{1.602 \times 10^{-19}\text{ J/eV}} = 5.11 \times 10^5\text{ eV}$$

or

$$E = 0.511\text{ MeV}$$ **rest energy** for the electron

The mass of the particle is often expressed with the same number thus:

$$m = \frac{E}{c^2} = 0.511\ \text{MeV}/c^2 \qquad \text{mass for the electron}$$

Now, applying the above to the muons produced by the cosmic rays, each has a total energy E given by

$$E = \gamma mc^2 = \frac{1}{\sqrt{1 - (0.998c)^2/c^2}} \times 105.7 \frac{\text{MeV}}{c^2} \times c^2$$
$$E = 1670\ \text{MeV}$$

and a measured mass (see Equation 2-5) of

$$\gamma m = E/c^2 = 1670\ \text{MeV}/c^2$$

This dependence of the measured mass on the speed of the particle has been verified by numerous experiments. Figure 2-7 illustrates a few of those results.

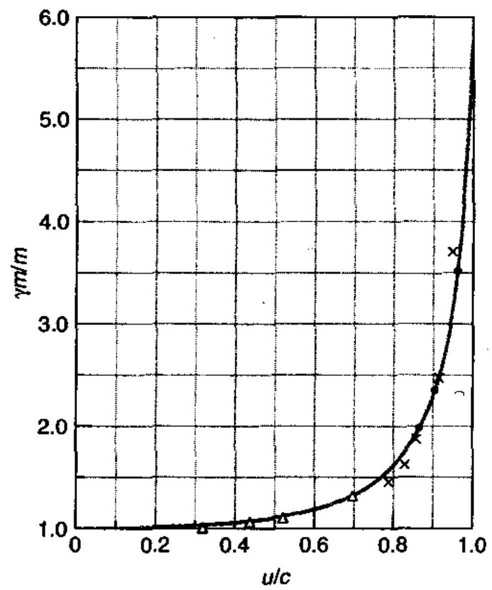


Fig. 2-7 A few of the many experimental measurements of the mass of electrons as a function of their speed u/c. The data points are plotted onto Equation 2-5, the solid line. The data points represent the work of Kaufmann (×, 1901), Bucherer (Δ, 1908), and Bertozzi (● 1964). Note that Kaufmann's work preceded the appearance of Einstein's 1905 paper on special relativity. [*Adapted from Figure 3-4 in R. Resnick and D. Halliday,* Basic Concepts in Relativity and Early Quantum Theory, *2d ed. (New York: Macmillan, 1992).*]

EXAMPLE 2-6 Change in the Solar Mass Compute the rate at which the sun is losing mass, given that the mean radius R of Earth's orbit is 1.50×10^8 km and the intensity of solar radiation at Earth (called the *solar constant*) is $1.36 \times 10^3\ \text{W/m}^2$.

Solution

1. The conversion of mass into energy, a consequence of conservation of energy in relativity, is implied by Equation 2-10. With $u = 0$ that equation becomes:

$$E = mc^2$$

2. Assuming that the sun radiates uniformly over a sphere of radius R, the total power P radiated by the sun is given by:

$$\begin{aligned} P &= \text{(area of the sphere)(solar constant)} \\ &= (4\pi R^2)(1.36 \times 10^3\ \text{W/m}^2) \\ &= 4\pi(1.50 \times 10^{11}\ \text{m})^2(1.36 \times 10^3\ \text{W/m}^2) \\ &= 3.85 \times 10^{26}\ \text{J/s} \end{aligned}$$

3. Thus, every second the sun emits 3.85×10^{26} J, which, from Equation 2-10, is the result of converting an amount of mass m given by:

$$\begin{aligned} m &= E/c^2 \\ &= \frac{3.85 \times 10^{26}\ \text{J}}{(3.00 \times 10^8\ \text{m/s})^2} \\ &= 4.3 \times 10^9\ \text{kg} \end{aligned}$$

Remarks: *Thus, the sun is losing 4.3×10^9 kg of mass (about 4 million metric tons) every second! If this rate of mass loss were to remain constant (which it will for the next few billion years), the sun's present mass of about 2.0×10^{30} kg would last "only" for about 10^{13} more years!*

$\tau_{universe} > 10^{16}$ yr

Exploring

Another Surprise!

One consequence of the fact that Newton's second law $\mathbf{F} = m\mathbf{a}$ is not relativistically invariant is yet another surprise—the lever paradox. Consider a lever of mass m at rest in S (see Figure 2-8). Since the lever is at rest, the net torque τ_{net} due to the forces $\mathbf{F}_x$ and $\mathbf{F}_y$ is zero, i.e. (using magnitudes):

$$\tau_{net} = \tau_x + \tau_y = -F_x b + F_y a = 0$$

and, therefore,

$$F_x b = F_y a$$

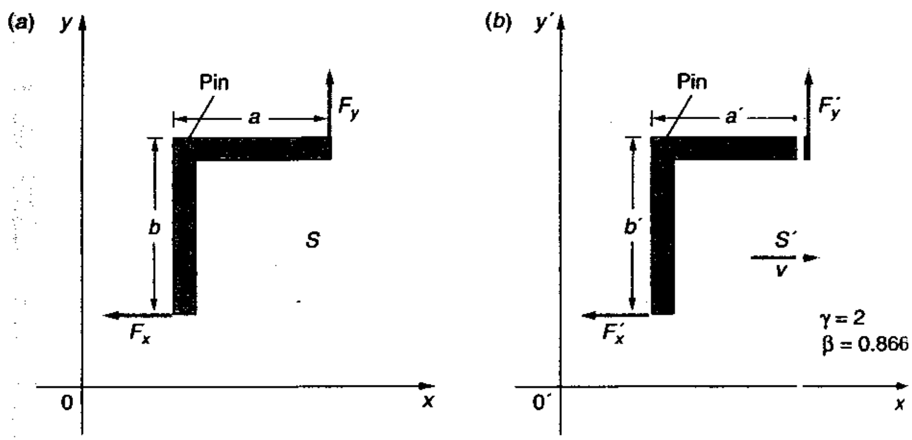


Fig. 2-8 (*a*) A lever in the *xy* plane of system *S* is free to rotate about the pin *P*, but is held at rest by the two forces F_x and F_y. (*b*) The same lever as seen by an observer in a S' which is moving with instantaneous speed *v* in the $+x$ direction. For the S' observer, the lever is moving in the $-x'$ direction.

An observer in system S' moving with $\beta = 0.866$ ($\gamma = 2$) with respect to S sees the lever moving in the $-x'$ direction and measures the torque to be

$$\begin{aligned}\tau'_{\text{net}} &= \tau'_x + \tau'_y = -F'_x b' + F'_y a' = -F_x b + (F_y/2)(a/2)\\ &= -F_x b + F_x b/4 = -(3/4)F_x b \neq 0\end{aligned}$$

where $F'_x = F_x$ and $F'_y = F_y/2$ (see Problem 2-52) and the lever is rotating!

The resolution of the paradox was first given by the German physicist Max von Laue (1879–1960). Recall that the net torque is the rate of change of the angular momentum **L**. The S' observer measures the work done per unit time by the two forces as

For F'_x: $\quad -F'_x v = -F_x v$

For F'_y: $\quad$ zero, since F'_y is perpendicular to the motion

and the change in mass Δm per unit time of the moving lever as

$$\frac{\Delta m}{\Delta t'} = \frac{\Delta E/c^2}{\Delta t'} = \frac{1}{c^2}\frac{\Delta E}{\Delta t'} = -\frac{1}{c^2}F_x v$$

The S' observer measures a change in the magnitude of the angular momentum per unit time given by

$$\tau_{\text{net}} = \frac{\Delta L'}{\Delta t'} = \frac{b\Delta p'}{\Delta t'} = \frac{bv\Delta m}{\Delta t'}$$

Substituting for $\Delta m/\Delta t'$ from above yields

$$\tau_{\text{net}} = \frac{\Delta L'}{\Delta t'} = bv\frac{-F_x v}{c^2} = -bF_x\frac{v^2}{c^2} = -bF_x\beta^2 = -\frac{3}{4}F_x b$$

As a result of the motion of the lever relative to S', an observer in that system sees the force F_x' doing net work on the lever, thus changing its angular momentum over time and the paradox vanishes.

The authors thank Costas Efthimiou for bringing this paradox to our attention.

2-3 Mass/Energy Conversion and Binding Energy

The identification of the term mc^2 as rest energy is not merely a convenience. Whenever additional energy ΔE in any form is stored in an object, the mass of the object is increased by $\Delta E/c^2$. This is of particular importance whenever we want to compare the mass of an object that can be broken into constituent parts with the mass of the parts (for example, an atom containing a nucleus and electrons, or a nucleus containing protons and neutrons). In the case of the atom, the mass changes are usually negligibly small (see Example 2-8). However, the difference between the mass of a *nucleus* and that of its constituent parts (protons and neutrons) is often of great importance.

As an example, consider Figure 2-9*a*, in which two particles, each with mass m, are moving toward each other, with speeds u. They collide with a spring that compresses and locks shut. (The spring is merely a device for visualizing energy storage.) In the Newtonian mechanics description, the original kinetic energy $E_k = 2(\frac{1}{2}mu^2)$ is converted into potential energy of the spring U. When the spring is unlocked, the potential energy reappears as kinetic energy of the particles. In relativity theory, the internal energy of the system, $E_k = U$, appears as an increase in rest mass of the system. That is, the mass of the system M is now greater than $2m$ by E_k/c^2. (We shall derive this result in the next example.) This change in mass is too small to be observed for ordinary-sized masses and springs, but it is easily observed in transformations that involve nuclei. For example, in the fission of a ^{235}U nucleus, the energy released as kinetic energy of the fission fragments is an appreciable fraction of the rest energy of the original nucleus. (See Example 12-4.) This energy can be calculated by measuring the difference between the mass of

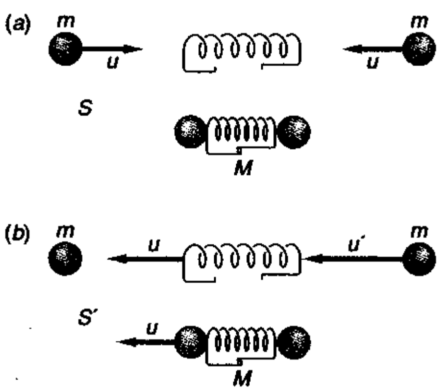


Fig. 2-9 Two objects colliding with a massless spring that locks shut. The total rest mass of the system M is greater than that of the parts $2m$ by the amount E_k/c^2, where E_k is the internal energy, which in this case is the original kinetic energy. (*a*) The event as seen in a reference frame S in which the final mass M is at rest. (*b*) The same event as seen in a frame S' moving to the right at speed u relative to S, so that one of the initial masses is at rest.

the original system and the total mass of the fragments. Einstein was the first to point out this possibility in 1905, even before the discovery of the atomic nucleus, at the end of a very short paper that followed his famous article on relativity.[7] After deriving the theoretical equivalence of energy and mass, he wrote:

Thomson 1880?

> It is not impossible that with bodies whose energy content is variable to a high degree (e.g., with radium salts) the theory may be successfully put to the test.

The relativistic conversion of mass into energy is the fundamental energy source in the nuclear-reactor-based systems that produce electricity in 30 nations and in large naval vessels and nuclear submarines.

EXAMPLE 2-7 Change in Rest Mass of the Two-Particle and Spring System of Figure 2-9 Derive the increase in the rest mass of a system of two particles in a totally inelastic collision. Let m be the mass of each particle so that the total mass of the system is $2m$ when the particles are at rest and far apart, and let M be the rest mass of the system when it has internal energy E_k. The original kinetic energy in the reference frame S (Figure 2-9*a*) is

$$E_k = 2mc^2(\gamma - 1) \qquad \textbf{2-25}$$

Solution

In a perfectly inelastic collision, momentum conservation implies that both particles are at rest after the collision in this frame, which is the center-of-mass frame. The total kinetic energy is therefore lost. We wish to show that if momentum is to be conserved in any reference frame moving with a constant velocity relative to S, the total mass of the system must increase by Δm, given by

$$\Delta m = \frac{E_k}{c^2} = 2m(\gamma - 1) \qquad \textbf{2-26}$$

We therefore wish to show that the total mass of the system with internal energy is M, given by

$$M = 2m + \Delta m = 2\gamma m \qquad \textbf{2-27}$$

To simplify the mathematics, we choose a second reference frame S' moving to the right with speed $v = u$ relative to frame S so that one of the particles is initially at rest, as shown in Figure 2-9*b*. The initial speed of the other particle in this frame is

$$u' = \frac{u - v}{1 - uv/c^2} = \frac{-2u}{1 + u^2/c^2} \qquad \textbf{2-28}$$

After collision, the particles move together with speed u toward the left (since they are at rest in S). The initial momentum in S' is

$$p_i' = \frac{mu'}{\sqrt{1 - u'^2/c^2}} \qquad \text{to the left}$$

The final momentum is

$$p_f' = \frac{Mu}{\sqrt{1 - u^2/c^2}} \qquad \text{to the left}$$

Using Equation 2-28 for u', squaring, dividing by c^2, and adding -1 to both sides give

$$1 - \frac{u'^2}{c^2} = 1 - \frac{4u^2/c^2}{(1 + u^2/c^2)^2} = \frac{(1 - u^2/c^2)^2}{(1 + u^2/c^2)^2}$$

Then

$$p_i' = \frac{m[2u/(1 + u^2/c^2)]}{(1 - u^2/c^2)/(1 + u^2/c^2)} = \frac{2mu}{1 - u^2/c^2}$$

Conservation of momentum in frame S' requires that $p_f' = p_i'$, or

$$\frac{Mu}{\sqrt{1 - u^2/c^2}} = \frac{2mu}{1 - u^2/c^2}$$

Solving for M we obtain

$$M = \frac{2m}{\sqrt{1 - u^2/c^2}} = 2\,\gamma m$$

which is Equation 2-27. Thus, the measured value of M would be $2\gamma m$.

If the latch in Figure 2-9*b* were to come unhooked suddenly, the two particles would fly apart with equal momenta, converting the rest mass Δm back into kinetic energy. The derivation is similar to that in Example 2-7.

Mass and Binding Energy

When a system of particles is held together by attractive forces, energy is required to break up the system and separate the particles. The magnitude of this energy E_b is called the ***binding energy*** of the system. An important result of the theory of special relativity which we shall illustrate by example in this section is:

> The mass of a bound system is less than that of the separated particles by E_b/c^2, where E_b is the binding energy.

In atomic and nuclear physics, masses and energies are typically given in atomic mass units (u) and electron volts (eV) rather than in the standard SI units of kilograms and joules. The u is related to the corresponding SI units by

$$1\text{u} = 1.66054 \times 10^{-27}\,\text{kg} = 931.5\ \text{MeV}/c^2 \qquad \textbf{2-29}$$

(The eV was defined in terms of the joule in Equation 2-24.) The rest energies of some elementary particles and a few light nuclei are given in Table 2-1, from which you can see by comparing the sums of the masses of the constituent particles with the nuclei listed that the mass of a nucleus is not the same as the sum of the masses of its parts.

The simplest example of nuclear binding energy is that of the deuteron ^{2}H, which consists of a neutron and a proton bound together. Its rest energy is 1875.61 MeV. The sum of the rest energies of the proton and neutron is 938.27 + 939.57 = 1877.84 MeV.

TABLE 2-1 Rest energies of some elementary particles and light nuclei

Particle	Symbol	Rest energy (MeV)
Photon	γ	0
Neutrino (antineutrino)†	$\nu(\bar{\nu})$	$< 2.8 \times 10^{-6}$
Electron (positron)	e or e^- (e^+)	0.5110
Muon	$\mu^{\pm}$	105.7
Pi meson	π^0	135
	$\pi^{\pm}$	139.6
Proton	p	938.272
Neutron	n	939.565
Deuteron	^{2}H or d	1875.613
Helion	^{3}He or h	2808.391
Alpha	^{4}He or α	3727.379

† *As we will discuss in Chapter 13, there are theoretical reasons and increasingly strong experimental evidence for the electron neutrinos to have a nonzero mass. There are other types of neutrinos whose masses may be as large as several MeV/c^2.*

Since this is greater than the rest energy of the deuteron, the deuteron cannot *spontaneously* break up into a neutron and a proton without violating conservation of energy. The binding energy of the deuteron is 1877.85 − 1875.63 = 2.22 MeV. In order to break up the deuteron into a proton and a neutron, at least 2.22 MeV must be added. This can be done by bombarding deuterons with energetic particles or electromagnetic radiation. If a deuteron is formed by combination of a neutron and a proton, the same amount of energy must be released.

EXAMPLE 2-8 Binding Energy of the Hydrogen Atom **The binding energies of the atomic electrons to the nuclei of atoms are typically of the order of 10^{-6} times those characteristic of particles in nuclei; consequently, the mass differences are correspondingly smaller. The binding energy of the hydrogen atom (the energy needed to remove the electron from the atom) is 13.6 eV. How much mass is lost when an electron and a proton form a hydrogen atom?**

Solution

The mass of a proton plus that of an electron must be greater than that of the hydrogen atom by

$$\frac{13.6 \text{ eV}}{931.5 \text{ MeV/u}} = 1.46 \times 10^{-8} \text{ u}$$

The mass difference is so small that it is usually neglected.

2-4 Invariant Mass

In Chapter 1 we discovered that, as a consequence of Einstein's relativity postulates, the coordinates for space and time are linearly dependent on one another in the Lorentz transformation that connects measurements made in different inertial reference frames. Thus, the time t became a coordinate, in addition to the space coordinates x, y, and z, in the four-dimensional relativistic "world" that we call spacetime. We noted in passing that the geometry of spacetime was not the familiar Euclidean geometry of our three-dimensional world, but the four-dimensional Lorentzian geometry. The difference became apparent when one compared the computation of the distance r between two points in space with that of the interval between two events in spacetime. The former is, of course, a vector **r** whose magnitude is given by $r^2 = x^2 + y^2 + z^2$. The vector **r** is unchanged (invariant) under a Galilean transformation in space, and quantities that transform like **r** are also vectors. The latter we called the spacetime interval Δs, and its magnitude, as we have seen, is given by

$$(\Delta s)^2 = (c\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2] \qquad \textbf{2-30}$$

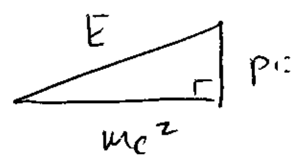
The interval Δs is the four-dimensional analog of **r** and, therefore, is called a *four-vector.* Just as x, y, and z are the components of the three-vector **r**, the components of the four-vector Δs are Δx, Δy, Δz, and $c\Delta t$. We have seen that Δs is also invariant under a Lorentz transformation in spacetime. Correspondingly, any quantity that transforms like Δs—i.e., is invariant under a Lorentz transformation—will also be a four-vector. The physical significance of the invariant interval Δs is quite profound: for timelike intervals, $\Delta s/c = \tau$ (the proper time interval); for spacelike intervals, $\Delta s = L_p$ (the proper length); and the intervals could be found from measurements made in *any* inertial frame.[8]

In the relativistic energy and momentum we have components of another four-vector. In the preceding sections we saw that momentum and energy, defined by Equations 2-6 and 2-10, respectively, were not only both conserved in relativity, but also together satisfied the Lorentz transformation, Equations 2-16 and 2-17, with the components of the momentum $\mathbf{p}(p_x, p_y, p_z)$ transforming like the space components of $\mathbf{r}(x, y, z)$ and the energy transforming like the time t. The questions then, are, What invariant four-vector are they components of? and, What is its physical significance? The answers to both turn out to be easy to find and yield for us yet another relativistic surprise. By squaring Equations 2-6 and 2-10, you can readily verify that

$$E^2 = (pc)^2 + (mc^2)^2 \qquad \textbf{2-31}$$

This very useful relation we will rearrange slightly to

$$(mc^2)^2 = E^2 - (pc)^2 \qquad \textbf{2-32}$$



Comparing the form of Equation 2-32 with that of Equation 2-30 and knowing that E and **p** transform according to the Lorentz transformation, we see that the magnitude of the invariant energy/momentum four-vector is the rest energy of the mass m! Thus, observers in all inertial frames will measure the same value for the rest energy of isolated systems and, since c is constant, the same value for the mass. Note that only in the rest frame of the mass m, i.e., the frame where $\mathbf{p} = 0$, are the rest energy and the total energy equal. Even though we have written Equation 2-31 for a single particle, we could as well have written the equations for momentum and energy in terms of the total momentum and total energy of an entire ensemble of noninteracting particles with arbitrary velocities. We would only need to write down Equations 2-6 and 2-10 for each particle and add them together. Thus, the Lorentz transformation for momentum and energy, Equations 2-16 and 2-17, holds for any system of particles, and so, therefore, does the invariance of the rest energy expressed by Equation 2-32.

We may state all of this more formally by saying that the *kinematic* state of the system is described by the four-vector Δs where

$$(\Delta s)^2 = (c\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]$$

and its *dynamic* state is described by the energy/momentum four-vector mc^2, given by

$$(mc^2)^2 = E^2 - (pc)^2$$

The next example illustrates how this works.

EXAMPLE 2-9 Rest Mass of a Moving Object A particular object is observed to move through the laboratory at high speed. Its total energy and the components of its momentum are measured by lab workers to be (in SI units) $E = 4.5 \times 10^{17}$ J, $p_x = 3.8 \times 10^8$ kg·m/s, $p_y = 3.0 \times 10^8$ kg·m/s, and $p_z = 3.0 \times 10^8$ kg·m/s. What is the object's rest mass?

Solution A

From Equation 2-32 we can write

$$\begin{aligned}(mc^2)^2 &= (4.5 \times 10^{17})^2 - [(3.8 \times 10^8 c)^2 + (3.0 \times 10^8 c)^2 + (3.0 \times 10^8 c)^2] \\ &= (4.5 \times 10^{17})^2 - [1.4 \times 10^{17} + 9.0 \times 10^{16} + 9.0 \times 10^{16}] c^2 \\ &= 2.0 \times 10^{35} - 2.9 \times 10^{34} \\ &= 1.74 \times 10^{35} \\ m &= (1.74 \times 10^{35})^{1/2}/c^2 = 4.6 \text{ kg}\end{aligned}$$

Solution B

A slightly different but sometimes more convenient calculation that doesn't involve carrying along large exponents makes use of Equation 2-32 divided by c^4:

$$m^2 = \left(\frac{E}{c^2}\right)^2 - \left(\frac{p}{c}\right)^2 \qquad \textbf{2-33}$$

Notice that this is simply a unit conversion, expressing each term in $(\text{mass})^2$ units—e.g., kg^2 when E and p are in SI units:

$$\begin{aligned}m^2 &= \left(\frac{4.5 \times 10^{17}}{c^2}\right)^2 - \left[\left(\frac{3.8 \times 10^8}{c}\right)^2 + \left(\frac{3.0 \times 10^8}{c}\right)^2 + \left(\frac{3.0 \times 10^8}{c}\right)^2\right] \\ &= (5.0)^2 - [(1.25)^2 + (1.0)^2 + (1.0)^2] \\ &= 25 - 3.56 \\ m &= (21.4)^{1/2} = 4.6 \text{ kg}\end{aligned}$$

In the example, we determined the rest energy and mass of a rapidly moving object using measurements made in the laboratory without the need to be in the system in which the object is at rest. This ability is of enormous benefit to nuclear, particle, and astrophysicists whose work regularly involves particles moving at speeds close to that of light. For particles or objects whose rest mass is known, we can use the invariant magnitude of the energy/momentum four-vector to determine the values of other dynamic variables, as illustrated in the next example.

EXAMPLE 2-10 Speed of a Fast Electron The total energy of an electron produced in a particular nuclear reaction is measured to be 2.40 MeV. Find the electron's momentum and speed in the laboratory frame. (The rest mass of the electron is 9.11×10^{-31}kg and its rest energy is 0.511 MeV.)

Solution

The magnitude of the momentum follows immediately from Equation 2-31:

$$pc = \sqrt{E^2 - (mc^2)^2} = \sqrt{(2.40\text{ MeV})^2 - (0.511\text{ MeV})^2}$$
$$= 2.34\text{ MeV}$$
$$p = 2.34\text{ MeV}/c$$

where we have again made use of the convenience of the eV as an energy unit. The resulting momentum unit MeV/c can be readily converted to SI units by converting the MeV to joules and dividing by c, i.e.,

$$1\text{ MeV}/c = \frac{1.60 \times 10^{-13}\text{ J}}{2.998 \times 10^8\text{ m/s}} = 5.34 \times 10^{-22}\text{ kg}\cdot\text{m/s}$$

Therefore, the conversion to SI units is easily done, if desired, and yields

$$p = 2.34\text{ MeV}/c \times \frac{5.34 \times 10^{-22}\text{ kg}\cdot\text{m/s}}{1\text{ MeV}/c}$$
$$p = 1.25 \times 10^{-21}\text{ kg}\cdot\text{m/s}$$

The speed of the particle is obtained by noting from Equation 2-32 or from Equations 2-6 and 2-10 that

$$\frac{u}{c} = \frac{pc}{E} = \frac{2.34\text{ MeV}}{2.40\text{ MeV}} = 0.975 \qquad \textbf{2-34}$$

or

$$u = 0.975c$$

It is extremely important to recognize that the invariant rest energy in Equation 2-32 is that of the *system* and that its value is *not* the sum of the rest energies of the particles of which the system is formed, if the particles move relative to one another. Earlier we used numerical examples of the binding energy of atoms and nuclei that illustrated this

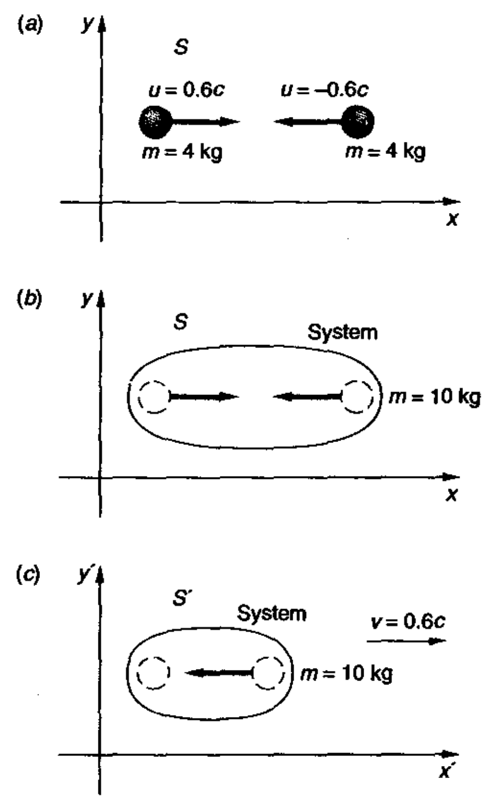


Fig. 2-10 (*a*) Two identical particles with rest mass 4 kg approach each other with equal but oppositely directed momenta. The rest mass of the system made up of the two particles is not 4 kg +4 kg, because the system's rest mass includes the mass equivalent of its internal motions. That value, 10 kg (*b*), would be the result of a measurement of the system's mass made by an observer in *S*, for whom the system is at rest, or by observers in any other inertial frames. (*c*) Transforming to *S*′ moving at $v = 0.6c$ with respect to *S*, as described in Example 2-11, also yields $m = 10$ kg.

fact by showing that the masses of atoms and nuclei were less than the sum of the masses of their constituents by an amount Δmc^2 that equaled the observed binding energy, but those were systems of interacting particles—i.e., there were forces acting between the constituents. A difference exists, even when the particles do not interact. To see this, let us focus our attention on specifically *what* mass is invariant.

Consider two identical noninteracting particles, each of rest mass $m = 4$ kg, moving toward each other along the *x* axis of *S* with momentum of magnitude $p_x = 3c \cdot$ kg, as illustrated in Figure 2-10*a*. The energy of each particle, using Equation 2-33, is

$$E/c^2 = \sqrt{m^2 + (p/c)^2} = \sqrt{(4)^2 + (3)^2} = 5 \text{ kg}$$

Thus, the total energy of the system is $5c^2 + 5c^2 = 10\ c^2 \cdot \text{kg}$, since the energy is a scalar. Similarly, the total momentum of the system is $3c - 3c = 0$, since the momentum is a vector and the momenta are equal and opposite. The rest mass of the system is then

$$m = \sqrt{(E/c^2)^2 - (p/c)^2} = \sqrt{(10)^2 - (0)^2} = 10 \text{ kg}$$

Thus, the system mass of 10 kg is *greater* than the sum of the masses of the two particles, 8 kg. (This is in contrast to bound systems, such as atoms, where the system mass is *smaller* than the total of the constituents.) This difference is not binding energy, since the particles are noninteracting. Neither does the 2-kg "mass difference" reside equally with the two particles. In fact it doesn't reside in any particular place, but is a property of the entire system. The correct interpretation is that the mass *of the system* is 10 kg.

While the invariance of the energy/momentum four-vector guarantees that observers in other inertial frames will also measure 10 kg as the mass of this system, let us allow for a skeptic or two and transform to another system S', e.g., the one shown in Figure 2-10*c*, just to be sure. This transformation is examined in the next example.

EXAMPLE 2-11 Lorentz Transformation of System Mass **For the system illustrated in Figure 2-10, show that an observer in S', which moves relative to S at $\beta = 0.6$, also measures the mass of the system to be 10 kg.**

Solution

1. The mass m measured in S' is given by Equation 2-33, which in this case is:

$$m = [(E'/c^2) - (p'_x/c)^2]^{1/2}$$

2. E' is given by Equation 2-16:

$$\begin{aligned} E' &= \gamma(E - vp_x) \\ &= \frac{1}{\sqrt{1-(0.6)^2}}(10c^2 - 0.6c \times 0) \\ &= (1.25)(10c^2) \\ &= 12.5\, c^2\cdot\text{kg} \end{aligned}$$

3. p'_x is also given by Equation 2-16:

$$\begin{aligned} p'_x &= \gamma(p_x - vE/c^2) \\ &= (1.25)[0 - (0.6c)(10c^2)/c^2] \\ &= -7.5\, c\cdot\text{kg} \end{aligned}$$

4. Substituting E' and p'_x into Equation 2-33 yields:

$$\begin{aligned} m &= [(12.5c^2/c^2)^2 - (-7.5c/c)^2]^{1/2} \\ &= [(12.5)^2 - (-7.5)^2]^{1/2} \\ &= 10\text{ kg} \end{aligned}$$

Remarks: *This result agrees with the value measured in S. The speed of S′ chosen for this calculation, v = 0.6c, is convenient in that one of the particles making up the system is at rest in S′; however, that has no effect on the generality of the solution.*

Thus, we see that it is the rest energy of any isolated system that is invariant, whether that system be a single atom or the entire universe. And, based on our brief discussions thus far, we note that the system's rest energy may be greater than, equal to, or less than the sum of the rest energies of the constituents depending on their relative velocities and the detailed character of any interactions between them.

QUESTIONS

4. Suppose two loaded boxcars, each of mass $m = 50$ metric tons, roll toward each other on level track at identical speeds u, collide, and couple together. Discuss the mass of this system before and after the collision. What is the effect of the magnitude of u on your discussion?
5. In 1787 Count Rumford (1753–1814) tried unsuccessfully to measure an increase in the weight of a barrel of water when he increased its temperature from 29°F to 61°F. Explain why, relativistically, you would expect such an increase to occur, and outline an experiment that might, in principle, detect the change. Since Count Rumford preceded Einstein by about 100 years, why might he have been led to such a measurement?

Massless Particles

Equation 2-32 formally allows positive, negative, and zero values for $(mc^2)^2$, just as was the case for the spacetime interval $(\Delta s)^2$. We have been tacitly discussing positive cases thus far in this section; a discussion of possible negative cases we will defer until Chapter 13. Here we need to say something about the $mc^2 = 0$ possibility. Note first of all that the idea of zero rest mass has no analog in classical physics, since classically $E_k = mu^2/2$ and $\mathbf{p} = m\mathbf{u}$. If $m = 0$, then the momentum and kinetic energy are always zero, too, and the "particle" would seem to be nothing at all, experiencing no second-law forces, doing no work, and so forth. However, for $mc^2 = 0$ Equation 2-32 states that, in relativity

$$E = pc \qquad (\text{for } m = 0) \tag{2-35}$$

and, together with Equation 2-34, that $u = c$; i.e., a particle whose mass is zero moves at the speed of light. Similarly, a particle whose speed is measured to be c will have $m = 0$ and satisfy $E = pc$.

We must be careful, however, because Equation 2-32 was obtained from the relativistic definitions of E and $\mathbf{p}$,

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} \qquad \mathbf{p} = \gamma m\mathbf{u} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

As $u \rightarrow c$, $1/(1 - u^2/c^2)^{1/2} \rightarrow \infty$; however, since m is also approaching 0, the quantity γm, which is tending toward 0/0, can (and does) remain defined. Indeed, there is ample experimental evidence for the existence of particles with $mc^2 = 0$.

Current theories suggest the existence of three such particles. Perhaps the most important of these and the one thoroughly verified by experiments is the *photon*, or a particle of electromagnetic radiation (i.e., light). Classically, electromagnetic radiation was interpreted via Maxwell's equations as a wave phenomenon, its energy and momentum being distributed continuously throughout the space occupied by the wave. It was discovered around 1900 that the classical view of light required modification in certain situations, the change being a confinement of the energy and momentum of the radiation into many tiny packets or bundles, which were referred to as photons. Photons move at light speed, of course, and, as we have noted, are required by relativity to have $mc^2 = 0$. Recall that the spacetime interval Δs for light is also zero. Strictly speaking, of course, the second of Einstein's relativity postulates prevents a Lorentz transformation to the rest system of light, since light moves at c relative to all inertial frames. Consequently, the term *rest mass* has no operational meaning for light.

EXAMPLE 2-12 Rest Energy of a System of Photons Remember that the rest energy of a system of particles is not the sum of the rest energies of the individual particles, if they move relative to one another. This applies to photons, too! Suppose two photons, one with energy 5 MeV and the second with energy 2 MeV, approach each other along the x axis. What is the rest energy of this system?

Solution

The momentum of the 5-MeV photon is (from Equation 2-35) $p_x = 5$ MeV/c and that of the 2-MeV photon is $p_x = -2$ MeV/c. Thus, the energy of the system is $E = 5\text{ MeV} + 2\text{ MeV} = 7\text{ MeV}$ and its momentum is $p = 5\text{ MeV}/c - 2\text{ MeV}/c = 3\text{ MeV}/c$. From Equation 2-32 the system's rest energy is

$$mc^2 = \sqrt{(7\text{ MeV})^2 - (3\text{ MeV})^2} = 6.3\text{ MeV!!}$$

A second particle whose rest energy is zero is the *gluon.* This massless particle transmits, or carries, the strong interaction between *quarks,* which are the "building blocks" of all fundamental particles, including protons and neutrons. The existence of gluons is well established experimentally. We will discuss quarks and gluons further in Chapter 13. Finally, there are strong theoretical reasons to expect that gravity is transmitted by a massless particle called the *graviton,* which is related to gravity in much the same way that the photon is related to the electromagnetic field. Gravitons, too, move at speed c. While direct detection of the graviton is beyond our current and foreseeable experimental capabilities, major international cooperative experiments are currently under way to detect gravity waves. (See Section 2-5.)

Until recently a fourth particle, the *neutrino,* was also thought to have zero rest mass. However, accumulating experimental evidence collected by the Super-Kamiokande and SNO imaging neutrino detectors, among others, makes it nearly certain that neutrinos are not massless. We discuss neutrino mass and its implications further in Chapters 11 and 13.

Creation and Annihilation of Particles

The relativistic equivalence of mass and energy implies still another remarkable prediction which has no classical counterpart. As long as momentum and energy are conserved in the process,[9] elementary particles with mass can combine with their *antiparticles,* the masses of both being completely converted to energy in a process called *annihilation.* An example is that of an ordinary electron. An electron can orbit briefly with its antiparticle, called a *positron,*[10] but then the two unite, mutually annihilating and producing two or three photons. The two-photon version of this process is shown schematically in Figure 2-11. Positrons are produced naturally by cosmic

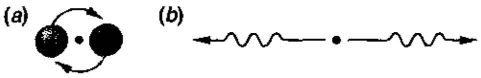
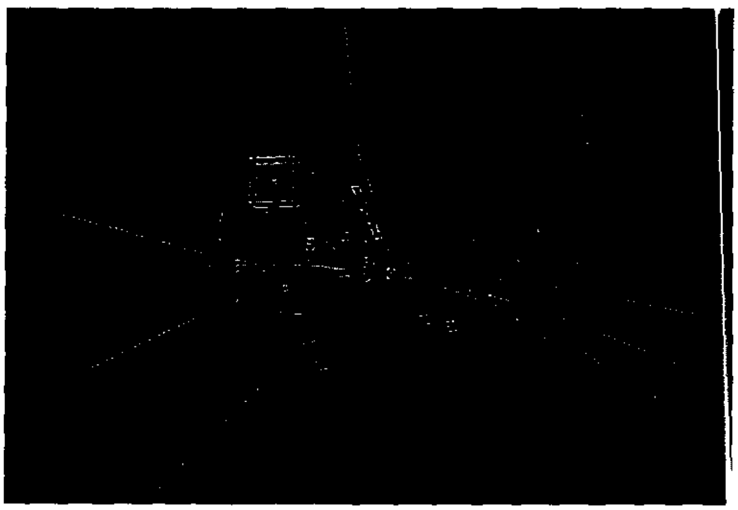


Fig. 2-11 (*a*) A positron orbits with an electron about their common center of mass, shown by the dot between them. (*b*) After a short time, typically of the order of 10^{-10} s for the case shown here, the two annihilate, producing two photons. The orbiting electron-positron pair, suggestive of a miniature hydrogen atom, is called *positronium*.

rays in the upper atmosphere and as the result of the decay of certain radioactive nuclei. P. A. M. Dirac had predicted their existence in 1928 while investigating the invariance of the energy/momentum four-vector.

If the speeds of both the electron and the positron $u \ll c$ (not a requirement for the process, but it makes the following calculation clearer), then the total energy of each particle $E \approx mc^2 = 0.511$ MeV. Therefore, the total energy of the system in Figure 2-11*a* before annihilation is $2mc^2 = 1.022$ MeV. Noting also from the diagram that the momenta of the particles are always opposite and equal, the total momentum of the system is zero. Conservation of momentum then requires that the total momentum of the two photons produced also be zero, i.e., that they move in opposite directions relative to the original center of mass and have equal momenta. Since $E = pc$ for photons, then they must also have equal energy. Conservation of energy then requires that the energy of each photon be 0.511 MeV. (Photons are usually called *gamma rays* when their energies are a few hundred keV or higher.) Notice from Example 2-12 that the magnitude of the energy/momentum four-vector (the rest energy) is not zero, even though both of the final particles are photons. In this case it equals the rest energy of the initial system. Analysis of the three-photon annihilation, although the calculation is a bit more involved, is similar.

By now it will not be a surprise to learn that the reverse process, the creation of mass from energy, can also occur under the proper circumstances. The conversion of mass and energy works both ways. The energy needed to create the new mass can be provided by the kinetic energy of another massive particle or by the "pure" energy of a photon. In either case, in determining what particles might be produced with a given amount of energy, it is important to be sure, as was the case with annihilation, that the appropriate conservation laws are satisfied. As we will discuss in detail in



Decay of a *Z* into an electron-positron pair in the UA1 detectors at CERN. This is the computer image of the first *Z* event recorded (30 April 1983). The newly created pair leaves the central detector in opposite directions at nearly the speed of light. [*CERN.*]

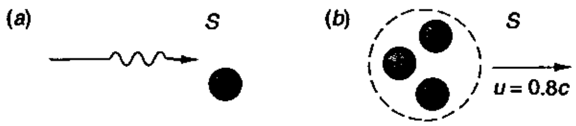


Fig. 2-12 (*a*) A photon of energy E and momentum $p = E/c$ encounters an electron at rest. The photon produces an electron-positron pair (*b*), and the group move off together at speed $u = 0.8c$.

Chapter 13, this restricts the creation process for certain kinds of particles (including electrons, protons, and neutrons) to producing only particle-antiparticle pairs. This means, for example, that the energy in a photon cannot be used to create a single electron, but must produce an electron-positron pair.

To see how the relativistic creation of mass goes, let us consider a particular situation, the creation of an electron-positron pair from the energy of a photon. The photon moving through space encounters, or "hits," an electron at rest in frame S as illustrated in Figure 2-12*a*.[11] Usually the photon simply scatters, but occasionally a pair is created. Encountering the existing electron is important, since it is not possible for the photon to produce spontaneously the two rest masses of the pair and also conserve momentum. (See Problem 2-44.) Some other particle must be nearby, not to provide energy to the creation process, but to acquire some of the photon's initial momentum. In this case we have selected an electron for this purpose, because it provides a neat example, but almost any particle would do. (See Example 2-13.)

While near the electron, the photon suddenly disappears, and an electron-positron pair appears. The process must occur very fast, since the photon, moving at speed c, will travel across a region as large as an atom in about 10^{-19} s. Let's suppose that the details of the interaction that produced this pair are such that the three particles all move off together toward the right in Figure 2-12*b* with the same speed u—i.e., they are all at rest in S', which moves to the right with speed u relative to S.[12] What must the energy E_γ of the photon be in order that this particular electron-positron pair be created? To answer this question, we first write the conservation of energy and momentum:

Before pair creation	**After pair creation**
$E_i = E_\gamma + mc^2$	$E_f = E_i = E_\gamma + mc^2$
$p_i = \dfrac{E_\gamma}{c}$	$p_f = p_i = \dfrac{E_\gamma}{c}$

where mc^2 = rest energy of an electron. In the final system after pair creation, the total rest energy is $3mc^2$ in this case. We know this because the invariant rest energy equals the sum of the rest energies of the *constituent* particles (the original electron and the pair) in the system where they do not move relative to one another, i.e., in S'. So in S we have for the system after pair creation:

$$(3mc^2)^2 = E^2 - (pc)^2$$

$$9(mc^2)^2 = (E_\gamma + mc^2)^2 - \left(\frac{E_\gamma c}{c}\right)^2$$

$$9(mc^2)^2 = E_\gamma^2 + 2E_\gamma mc^2 + (mc^2)^2 - E_\gamma^2$$

Noting that the E_γ^2 terms cancel, and dividing the remaining terms by mc^2, we see that

$$E_\gamma = 4mc^2$$

Thus, the initial photon needs energy equal to 4 electron rest energies in order to create 2 new electron rest masses in this case. Why is the "extra" energy needed? Because the three electrons in the final system share momentum E_γ/c, they must *also* have kinetic energy E_k given by

$$\begin{aligned} E_k &= E - 3mc^2 = (E_\gamma + mc^2) - 3mc^2 \\ &= 4mc^2 + mc^2 - 3mc^2 = 2mc^2 \end{aligned}$$

or the initial photon must provide the $2mc^2$ necessary to create the electron and positron masses and the additional $2mc^2$ of kinetic energy that they and the existing electron will share as a result of momentum conservation. The speed u at which the group of particles moves in S can be found from $u/c = pc/E$ (Equation 2-34):

$$u/c = \frac{\left(\dfrac{E_\gamma}{c} \times c\right)}{(E_\gamma + mc^2)} = \frac{4mc^2}{5mc^2} = 0.8$$

The portion of the incident photon's energy that is needed to provide kinetic energy in the final system is reduced if the mass of the existing particle is larger than that of an electron and, indeed, can be made negligibly small, as illustrated in the following example.

EXAMPLE 2-13 Threshold for Pair Production What is the minimum or threshold energy that a photon must have in order to produce an electron positron pair?

Solution

The energy E_γ of the initial photon must be

$$E_\gamma = mc^2 + E_{k-} + mc^2 + E_{k+} + E_{kM}$$

where mc^2 = electron rest energy, E_{k-} and E_{k+} are the kinetic energies of the electron and positron, respectively, and E_{kM} = kinetic energy of the existing particle of mass M. Since we are looking for the threshold energy, consider the limiting case where the pair are created at rest in S, i.e., $E_{k-} = E_{k+} = 0$ and correspondingly $p_- = p_+ = 0$. Therefore, momentum conservation requires that

$$p_{\text{initial}} = E_\gamma/c = p_{\text{final}} = \frac{Mu}{\sqrt{1 - u^2/c^2}}$$

where u = speed of recoil of the mass M. Since the masses of single atoms are in the range of 10^3 to 10^5 MeV/c^2 and the value of E_γ at the threshold is clearly less than about 2 MeV (i.e., it must be less than the value $E_\gamma = 4mc^2 = 2.044$ MeV), the speed with which M recoils from the creation event is quite small compared with c, even for the smallest M available, a single proton! (See Table 2-1.) Thus, the kinetic energy $E_{kM} \approx \frac{1}{2}Mu^2$ becomes negligible, and we conclude that the minimum energy E_γ of the initial photon that can produce an electron-positron pair is $2mc^2$, i.e., that needed just to create the two rest masses.

Some Useful Equations and Approximations

As we have seen, in relativistic dynamics it is most often the momentum or energy of a particle that is known rather than speed. We saw that Equation 2-6 for the relativistic momentum and Equation 2-10 for the relativistic energy could be combined to eliminate the speed u and yield the very useful relation

$$E^2 = (pc)^2 + (mc^2)^2 \qquad \textbf{2-31}$$

Extremely Relativistic Case The triangle shown in Figure 2-13 is sometimes useful in remembering this result. If the energy of a particle is much greater than its rest energy mc^2, the second term on the right of Equation 2-31 can be neglected, giving the useful approximation

$$E \approx pc \qquad \text{for } E >> mc^2 \qquad \textbf{2-36}$$

This approximation is accurate to about 1 percent or better if E is greater than about $8mc^2$. Equation 2-36 is the exact relation between energy and momentum for particles with zero rest mass.

From Equation 2-36 we see that the momentum of a high-energy particle is simply its total energy divided by c. A convenient unit of momentum is MeV/c. The momentum of a charged particle is usually determined by measuring the radius of curvature of the path of the particle moving in a magnetic field. If the particle has charge q and a velocity $\mathbf{u}$, it experiences a force in a magnetic field $\mathbf{B}$ given by

$$\mathbf{F} = q\mathbf{u} \times \mathbf{B}$$

where $\mathbf{F}$ is perpendicular to the plane formed by $\mathbf{u}$ and $\mathbf{B}$ and, hence, is always perpendicular to $\mathbf{u}$. Since the magnetic force is always perpendicular to the velocity, it does no work on the particle (the work-energy theorem also holds in relativity), so the energy of the particle is constant. From Equation 2-10 we see that if the energy is constant, γ must be a constant, and therefore the speed u is also constant. Therefore,

$$\mathbf{F} = q\mathbf{u} \times \mathbf{B} = \frac{d\mathbf{p}}{dt} = \frac{d(\gamma m\mathbf{u})}{dt} = \gamma m \frac{d\mathbf{u}}{dt}$$

For the case $\mathbf{u} \perp \mathbf{B}$, the particle moves in a circle with centripetal acceleration u^2/R. (If $\mathbf{u}$ is not perpendicular to $\mathbf{B}$, the path is a helix. Since the component of $\mathbf{u}$ parallel to $\mathbf{B}$ is unaffected, we shall consider only motion in a plane.) We then have

$$quB = m\gamma\left|\frac{d\mathbf{u}}{dt}\right| = m\gamma\left(\frac{u^2}{R}\right)$$

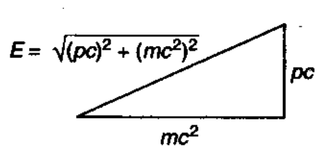


Fig. 2-13 Triangle showing the relation between energy, momentum, and rest mass in special relativity. *Caution:* Remember that E and pc are not relativistically invariant. The invariant is mc^2.

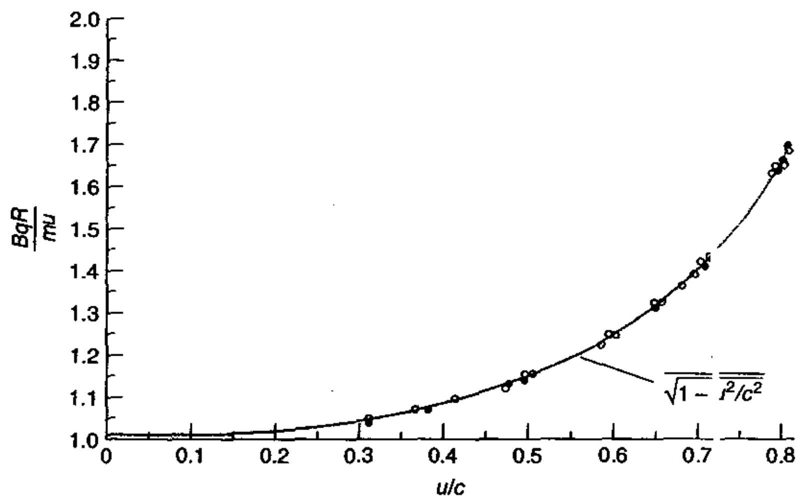


Fig. 2-14 *BqR/mu* versus *u/c* for particle of charge *q* and mass *m* moving in a circular orbit of radius *R* in a magnetic field *B*. The agreement of the data with the curve predicted by relativity theory supports the assumption that the force equals the time rate of change of relativistic momentum. [*Adapted from I. Kaplan,* Nuclear Physics, *2d ed. (Reading, Mass.: Addison-Wesley, 1962), by permission.*]

or

$$BqR = m\gamma u = p \qquad \textbf{2-37}$$

This is the same as the nonrelativistic expression except for the factor of γ. Figure 2–14 shows a plot of BqR/mu versus u/c. It is useful to rewrite Equation 2–37 in terms of practical but mixed units; the result is

$$p = 300\,BR\left(\frac{q}{e}\right) \qquad \textbf{2-38}$$

where p is in MeV/c, B is in tesla, and R is in meters.

EXAMPLE 2-14 Electron in a Magnetic Field **What is the approximate radius of the path of a 30-MeV electron moving in a magnetic field of 0.05 tesla (= 500 gauss)?**

Solution

1. The radius of the path is given by rearranging Equation 2-38 and substituting $q = e$:

$$R = \frac{p}{300B}$$

2. In this situation the total energy E is much greater than the rest energy mc^2:

$$E = 30\text{ MeV} \gg mc^2 = 0.511\text{ MeV}$$

3. Equation 2-36 may then be used to determine p:

$$p \approx E/c$$
$$= 30 \text{ MeV}/c$$

4. Substituting this approximation for p into Equation 2-38 yields:

$$R = \frac{30 \text{ MeV}/c}{(300)(0.05)}$$
$$= 2 \text{ m}$$

***Remarks:** In this case the error made by using the approximation, Equation 2-36, rather than the exact solution, Equation 2-31, is only about 0.01 percent.*

Nonrelativistic Case Nonrelativistic expressions for energy, momentum, and other quantities are often easier to use than the relativistic ones, so it is important to know when these expressions are accurate enough. As $\gamma \rightarrow 1$, all the relativistic expressions approach the classical ones. In most situations, the kinetic energy or total energy is given, so that the most convenient expression for calculating γ is, from Equation 2-10,

$$\gamma = \frac{E}{mc^2} = 1 + \frac{E_k}{mc^2} \qquad \textbf{2-39}$$

When the kinetic energy is much less than the rest energy, γ is approximately 1 and nonrelativistic equations can be used. For example, the classical approximation $E_k \approx \left(\frac{1}{2}\right)mu^2 = p^2/2m$ can be used instead of the relativistic expression $E_k = (\gamma - 1)mc^2$ if E_k is much less than mc^2. We can get an idea of the accuracy of these expressions by expanding γ, using the binomial expansion as was done in Section 2-2, and examining the first term that is *neglected* in the classical approximation. We have

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + \cdots$$

and

$$E_k = (\gamma - 1)mc^2 \approx \frac{1}{2}mu^2 + \frac{3}{2}\frac{\left(\frac{1}{2}mu^2\right)^2}{mc^2}$$

Then

$$\frac{E_k - \frac{1}{2}mu^2}{E_k} \approx \frac{3}{2}\frac{E_k}{mc^2}$$

For example, if $E_k/mc^2 \approx 1$ percent, the error in using the approximation $E_k \approx \left(\frac{1}{2}\right)mu^2$ is about 1.5 percent.

At very low energies, the velocity of a particle can be obtained from its kinetic energy $E_k \approx \left(\frac{1}{2}\right)mu^2$ just as in classical mechanics. At very high energies, the velocity of a particle is very near c. The following approximation is sometimes useful (see Problem 2-27):

$$\frac{u}{c} \approx 1 - \frac{1}{2\gamma^2} \qquad \text{for} \qquad \gamma \gg 1 \tag{2-40}$$

An exact expression for the velocity of a particle in terms of its energy and momentum was obtained in Example 2-10:

$$\frac{u}{c} = \frac{pc}{E} \tag{2-41}$$

This expression is, of course, not useful if the approximation $E \approx pc$ has already been made.

EXAMPLE 2-15 **Different Particles, Same Energy** An electron and a proton are each accelerated through 10×10^6 V. Find γ, the momentum, and the speed for each.

Solution

Since each particle has a charge of magnitude e, each acquires a kinetic energy of 10 MeV. This is much greater than the 0.511 MeV rest energy of the electron and much less than the 938.3 MeV rest energy of the proton. We shall calculate the momentum and speed of each particle exactly, and then by means of the nonrelativistic (proton) or the extreme relativistic (electron) approximations.

1. We first consider the electron. From Equation 2-39 we have

$$\gamma = 1 + \frac{E_k}{mc^2} = 1 + \frac{10\text{ MeV}}{0.511\text{ MeV}} = 20.57$$

Since the total energy is $E_k + mc^2 = 10.511$ MeV, we have, from the magnitude of the energy/momentum four-vector (Equation 2-31),

$$pc = \sqrt{E^2 - (mc^2)^2} = \sqrt{(10.51)^2 - (0.511)^2}$$
$$= 10.50\text{ MeV}$$

The exact calculation then gives $p = 10.50$ MeV/c. The high-energy or extreme relativistic approximation $p \approx E/c = 10.51$ MeV/c is in good agreement with the exact result. If we use Equation 2-34, we obtain for the speed $u/c = pc/E = 10.50$ MeV/10.51 MeV $= 0.999$. For comparison the approximation of Equation 2-40 gives

$$\frac{u}{c} \approx 1 - \frac{1}{2}\left(\frac{1}{\gamma}\right)^2 = 1 - \frac{1}{2}\left(\frac{1}{20.57}\right)^2 = 0.999$$

2. For the proton, the total energy is $E_k + mc^2 = 10$ MeV $+ 938.3$ MeV $=$ 948.3 MeV. From Equation 2-39 we obtain $\gamma = 1 + E_k/mc^2 = 1 + 10/938.3 = 1.01$. Equation 2-31 gives for the momentum

$$pc = \sqrt{E^2 - (mc^2)^2} = \sqrt{(948.3)^2 - (938.3)^2}$$
$$= 137.4 \text{ MeV}$$

The nonrelativistic approximation gives

$$E_k \approx \frac{1}{2}mu^2 = \frac{(mu)^2}{2m} \approx \frac{p^2}{2m} = \frac{p^2c^2}{2mc^2}$$

or

$$pc \approx \sqrt{2mc^2E_k} = \sqrt{(2)(938.3)(10)}$$
$$= 137.0 \text{ MeV}$$

The speed can be determined from Equation 2-34 exactly or from $p = mu$ approximately. From Equation 2-34 we obtain

$$\frac{u}{c} = \frac{pc}{E} = \frac{137.4}{948.3} = 0.1449$$

From $p \approx mu$, the nonrelativistic expression for p, we obtain

$$\frac{u}{c} \approx \frac{pc}{mc^2} = \frac{137.0}{938.3} = 0.1460$$

2-5 General Relativity

The generalization of relativity theory to noninertial reference frames by Einstein in 1916 is known as the *general theory of relativity.* This theory is much more difficult mathematically than the special theory of relativity, and there are fewer situations in which it can be tested. Nevertheless, its importance in the areas of astrophysics and cosmology, and the need to take account of its predictions in the design of such things as global navigation systems,[13] call for its inclusion here. A full description of the general theory uses tensor analysis at a quite sophisticated level, well beyond the scope of this book, so we will be limited to qualitative or, in some instances, semi-quantitative discussions. An additional purpose to the discussion that follows is to give you something that few people will ever have, namely, an acquaintance with one of the most remarkable of all scientific accomplishments and a bit of a feel for the man who did it.

Einstein's development of the general theory of relativity was not motivated by any experimental enigma. Instead, it grew out of his desire to include the descriptions of *all* natural phenomena within the framework of the special theory. By 1907 he realized that he could accomplish that goal with the single exception of the law of gravitation. About that exception he said,[14]

The exceptional sensitivity of modern electronic systems is such that general relativistic effects are included in the design of the Global Positioning System (GPS).

I felt a deep desire to understand the reason behind this [exception].

The "reason" came to him, as he said later, while he was sitting in a chair in the patent office in Bern. He described it like this:[15]

> Then there occurred to me the happiest thought of my life, in the following form. The gravitational field has only a relative existence in a way similar to the electric field generated by magnetoelectric induction. *Because for an observer falling freely from the roof of a house there exists*—at least in his immediate surroundings—*no gravitational field.* [Einstein's italics] . . . The observer then has the right to interpret his state as "at rest."

Out of this "happy thought" grew the *principle of equivalence* that became Einstein's fundamental postulate for general relativity.

Principle of Equivalence

The basis of the general theory of relativity is what we may call Einstein's third postulate, the principle of equivalence, which states:

> A homogeneous gravitational field is completely equivalent to a uniformly accelerated reference frame.

This principle arises in a somewhat different form in Newtonian mechanics because of the apparent identity of gravitational and inertial mass. In a uniform gravitational field, all objects fall with the same acceleration g independent of their mass because the gravitational force is proportional to the (gravitational) mass while the acceleration varies inversely with the (inertial) mass. That is, the mass m in

$$\mathbf{F} = m\mathbf{a} \qquad \text{(inertial } m\text{)}$$

and that in

$$\mathbf{F}_G = \frac{GMm}{r^2}\hat{\mathbf{r}} \qquad \text{(gravitational } m\text{)}$$

appear to be identical in classical mechanics, although classical theory provides no explanation for this equality. For example, near Earth's surface, $F_G = GMm/r^2 = m_{\text{grav}}\, g = m_{\text{inertial}}\, a = F$. Recent modern experiments have shown $m_{\text{inertial}} = m_{\text{grav}}$ to better than one part in 10^{12}.

To understand what the equivalence principle means, consider a compartment in space far away from any matter and undergoing uniform acceleration $\mathbf{a}$ as shown in Figure 2-15*a*. If people in the compartment drop objects, they fall to the "floor" with acceleration $\mathbf{g} = -\mathbf{a}$. If they stand on a spring scale, it will read their "weight" of magnitude ma. No mechanics experiment can be performed *within* the compartment that will distinguish whether the compartment is actually accelerating in space or is at rest (or moving with uniform velocity) in the presence of a uniform gravitational field $\mathbf{g} = -\mathbf{a}$.

Einstein broadened the principle of equivalence to apply to *all* physical experiments, not just to mechanics. In effect, he assumed that there is no experiment of any kind that can distinguish uniformly accelerated motion from the presence of a gravitational field. A direct consequence of the principle is that $m_{\text{grav}} = m_{\text{inertial}}$ is a requirement, not a coincidence. The principle of equivalence extends Einstein's first

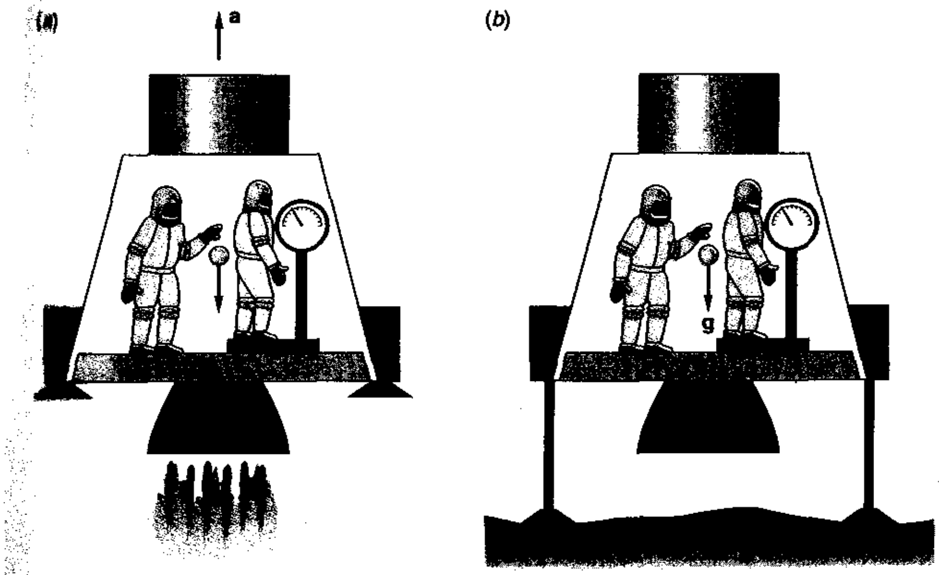


Fig. 2-15 Results from experiments in a uniformly accelerated reference frame (*a*) cannot be distinguished from those in a uniform gravitational field (*b*) if the acceleration **a** and gravitational field **g** have the same magnitude.

postulate, the principle of relativity, to *all* reference frames, noninertial (i.e., accelerated) as well as inertial. It follows that there is no absolute acceleration of a reference frame. Acceleration, like velocity, is only relative.

Question

6. For his 76th (and last) birthday Einstein received a present designed to demonstrate the principle of equivalence. It is shown in Figure 2-16. The object is, starting with the ball hanging down as shown, to put the ball into the cup with a method that works every time (as opposed to random shaking). How would you do it? (*Note*: When it was given to Einstein, he was delighted and did the experiment correctly immediately.)

Some Predictions of General Relativity

In his first paper on general relativity, in 1916, Einstein was able to explain quantitatively a discrepancy of long standing between the measured and (classically) computed values of the advance of the perihelion of Mercury's orbit, about 43 arc seconds/century. It was the first success of the new theory. A second prediction, the bending to light in a gravitational field, would seem to be more difficult to measure owing to the very small effect. However, it was accurately confirmed less than five years later when Arthur Eddington measured the deflection of starlight passing near the limb of the sun during a total solar eclipse. The theory also predicts the slowing of light itself and the slowing of clocks—i.e., frequencies—in gravitational fields, both of considerable importance to the determination of astronomical distances and stellar recession rates. The predicted slowing of clocks, called gravitational redshift, was demonstrated by

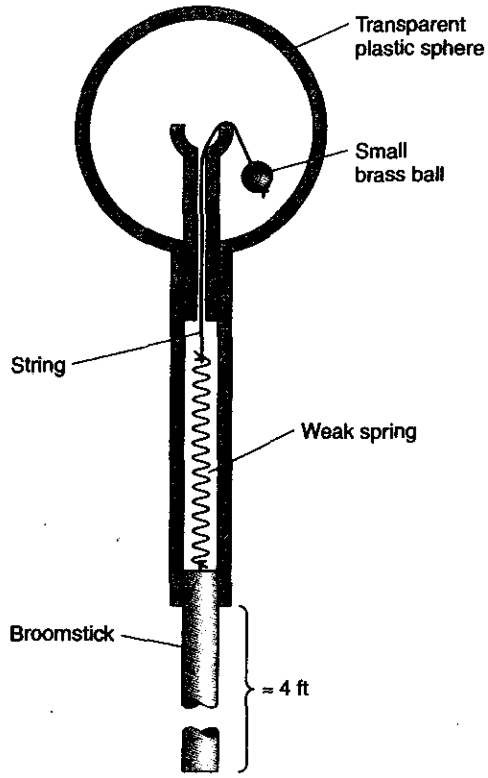


Fig. 2-16 Principle of equivalence demonstrator given to Einstein by E. M. Rogers. The object is to put the hanging brass ball into the cup by a technique that always works. The spring is weak, too weak to pull the ball in as it stands, and is stretched even when the ball is in the cup. The transparent sphere, about 10 cm in diameter, does not open. [*From A. P. French,* Albert Einstein: A Centenary Volume *(Cambridge, Mass.: Harvard University Press, 1979).*]

Pound and co-workers in 1960 in Earth's gravitational field using the ultrasensitive frequency measuring technique of the Mössbauer effect (see Chapter 11). The slowing of light was conclusively measured in 1971 by Shapiro and co-workers using radar signals reflected from several planets. Two of these experimental tests of relativity's predictions, bending of light and gravitational redshift, are discussed in the Exploring sections that follow. The perihelion of Mercury's orbit and the delay of light are discussed in More sections on the Web page. Many other predictions of general relativity are subjects of active current research. Two of these, black holes and gravity waves, are discussed briefly in the concluding paragraphs of this chapter.

This relativistic effect results in gravitational lenses in the cosmos that focus light from extremely distant galaxies, greatly improving their visibility in telescopes, both on Earth and in orbit.

Exploring

Deflection of Light in a Gravitational Field

With the advent of special relativity, several features of the Newtonian law of gravitation $F_G = GMm/r^2$ became conceptually troublesome. One of these was the implication from the relativistic concept of mass-energy equivalence that even particles with zero rest mass should exhibit properties like weight and inertia, thought of classically as masslike; classical theory does not include such particles. According to the equivalence principle, however, light, too, would experience the gravitational force. Indeed, the deflection of a light beam passing through the gravitational field near a large mass was one of the first consequences of the equivalence principle to be tested experimentally.

To see why a deflection of light would be expected, consider Figure 2-17, which shows a beam of light entering an accelerating compartment. Successive positions of the compartment are shown at equal time intervals. Because the compartment is accelerating, the distance it moves in each time interval increases with time. The path of the beam of light, as observed from within the compartment, is therefore a parabola. But according to the equivalence principle, there is no way to distinguish between an accelerating compartment and one with uniform velocity in a uniform gravitational field. We conclude, therefore, that a beam of light will accelerate in a gravitational field as do objects with rest mass. For example, near the surface of Earth, light will fall with acceleration 9.8 m/s^2. This is difficult to observe because of the enormous speed of light. For example, in a distance of 3000 km, which takes about 0.01 s to cover, a beam of light should fall about 0.5 mm. Einstein pointed out that the deflection of a light beam in a gravitational field might be observed when light from a distant star passes close to the sun.[16] The deflection, or bending, is computed as follows. Rewriting the spacetime interval Δs (Equation 1-32) in differential form and converting the space Cartesian coordinates to polar coordinates (in two space dimensions, since the deflection occurs in a plane) yields

$$ds^2 = c^2dt^2 - (dr^2 + r^2d\theta^2) \qquad \textbf{2-42}$$

Einstein showed that this expression is slightly modified in the presence of a (spherical, nonrotating) mass M to become

$$ds^2 = \gamma(r)^2c^2dt^2 - dr^2/\gamma(r)^2 - r^2d\theta^2 \qquad \textbf{2-43}$$

GRAVITATIONAL time dilation length contraction

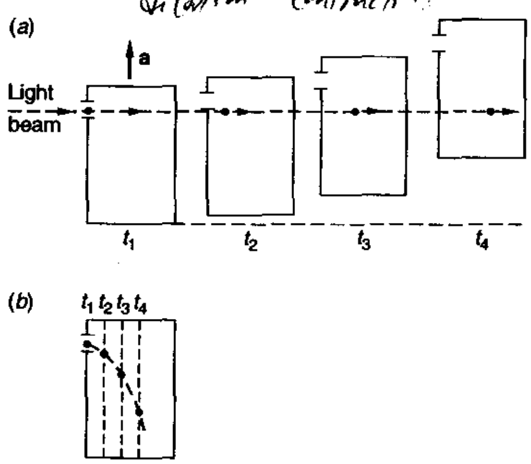


Fig. 2-17 (*a*) Light beam moving in a straight line through a compartment that is undergoing uniform acceleration. The position of the light beam is shown at equally spaced times t_1, t_2, t_3, t_4. (*b*) In the reference frame of the compartment, the light travels in a parabolic path, as would a ball were it projected horizontally. Note that in both (*a*) and (*b*) the vertical displacements are greatly exaggerated for emphasis.

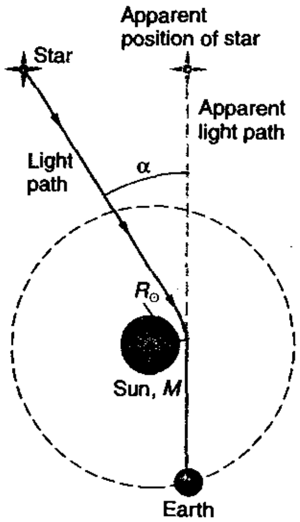


Fig. 2-18 Deflection (greatly exaggerated) of a beam of starlight due to the gravitational attraction of the sun.

where $\gamma(r) = (1 - 2GM/c^2r)^{1/2}$ with G = universal gravitational constant and r = distance from the center of mass M. The factor $\gamma(r)$ is roughly analogous to the γ of special relativity. In the following Exploring section on gravitational redshift, we describe how $\gamma(r)$ arises. For now, $\gamma(r)$ can be thought of as correcting for *gravitational time dilation* (the first term on the right of Equation 2-43) and *gravitational length contraction* (the second term).

This situation is illustrated in Figure 2-18, which shows the light from a distant star just grazing the edge of the sun. The gravitational deflection of the light (with mass $\gamma m = E/c^2$) can be treated as a refraction of the light. The speed of light is reduced to $\gamma(r)c$ in the vicinity of the mass M, since $\gamma(r) < 1$ (see Equation 2-43), thus bending the wave fronts, and hence the beam, toward M. This is analogous to the deflection of starlight toward Earth's surface as a result of the changing density—hence index of refraction—of the atmosphere. By integrating Equation 2-43 over the entire trajectory of the light beam (recall that $ds = 0$ for light) as it passes by M, the total deflection α is found to be[17]

$$\alpha = 4\,G\,M/c^2 R_\odot \qquad \textbf{2-44}$$

where $R_\odot$ = distance of closest approach of the beam to the center of M. For a beam just grazing the sun, $R_\odot$ = solar radius = 6.96×10^8 m. Substituting the values for G and the solar mass ($M = 1.99 \times 10^{30}$ kg) yields $\alpha = 1.75$ arc second.[18]

Ordinarily, of course, the brightness of the sun prevents astronomers (or anyone else) from seeing stars close to the limbs (edges) of the sun, except during a total eclipse. Einstein completed the calculation of α in 1915, and in 1919 expeditions were organized by Eddington[19] at two points along the line of totality of a solar eclipse, both of which were successful in making measurements of α for several stars and in testing the predicted $1/R_\odot$ dependence of α. The measured values of α for grazing beams at the two sites were:

At Sobral (South America):

$$\alpha = 1.98 \pm 0.12 \text{ arc seconds}$$

At Principe Island (Africa):

$$\alpha = 1.61 \pm 0.30 \text{ arc seconds}$$

their average agreeing with the general relativistic prediction to within about 2 percent. Figure 2-19 illustrates the agreement of the $1/R_\odot$ dependence with Equation 2-44. (Einstein learned of the successful measurements via a telegram from H. A. Lorentz.) Since 1919, many measurements of α have been made during eclipses. Since the development of radio telescopes, which are not blinded by sunlight and hence don't require a total eclipse, many more measurements have been made. The latest data agree with the deflection predicted by general relativity to within about 0.1 percent.

The gravitational deflection of light is being put to use by modern astronomers via the phenomenon of *gravitational lenses* to help in the study of galaxies and other large masses in space. Light from very distant stars and galaxies passing near or through other galaxies or clusters of galaxies between the source and Earth can be bent, or refracted, so as to reach Earth in much the same way that light from an object on a bench in the laboratory can be refracted by a glass lens and thus reach the eye of an observer. The intervening galaxy cluster can thus produce images of

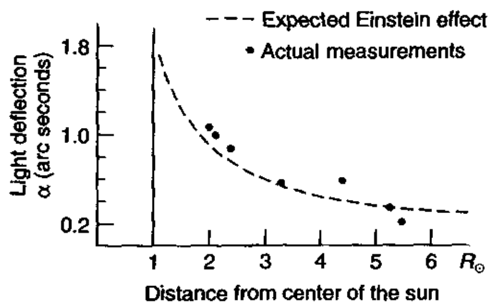


Fig. 2-19 The deflection angle α depends on the distance of closest approach $R_\odot$ according to Equation 2-44. Shown here is a sample of the data for 7 of the 13 stars measured by the Eddington expeditions. The agreement with the relativistic prediction is apparent.

the distant source, even magnified and distorted ones, just as the glass lens can. Figure 2-20*a* will serve as a reminder of a refracting lens in the laboratory, while Figure 2-20*b* illustrates the corresponding action of a gravitational lens. The accompanying photograph shows the images of several distant galaxies drawn out into arcs by the lens effect of the cluster of galaxies in the center. The first confirmed discovery of images formed by a gravitational lens was made in 1979 by D. Walsh and his co-workers. It was the double image of the quasar QSO 0957. Since then astronomers have found many such images. Their discovery and interpretation is currently an active area of research.

Exploring
Gravitational Redshift

A second prediction of general relativity concerns the rates of clocks and the frequencies of light in a gravitational field. As a specific case which illustrates the gravitational redshift as a direct consequence of the equivalence principle, suppose we consider two identical light sources (A and A') and detectors (B and B') located in identical spaceships (S and S') as illustrated in Figure 2-21 (page 113). The spaceship S' in Figure 2-21*b* is located far from any mass. At time $t = 0$, S' begins to accelerate, and simultaneously an atom in the source A' emits a light pulse of its characteristic frequency f_0. During the time t ($= h/c$) for the light to travel from A' to B', B' acquires a speed $v = at = gh/c$, and the detector at B', receding from the original location of A', measures the frequency of the incoming light to be f *redshifted* by a fractional amount $(f_0 - f)/f_0 \approx \beta$ for $v \ll c$. (See Section 1-5.) Thus,

$$(f_0 - f)/f_0 = \Delta f/f_0 \approx \beta = v/c = gh/c^2 \qquad \textbf{2-45}$$

Notice that the right side of Equation 2-45 is equal to the gravitational potential (i.e., the gravitational potential energy per unit mass) $\Delta\phi = gh$ between A and B,

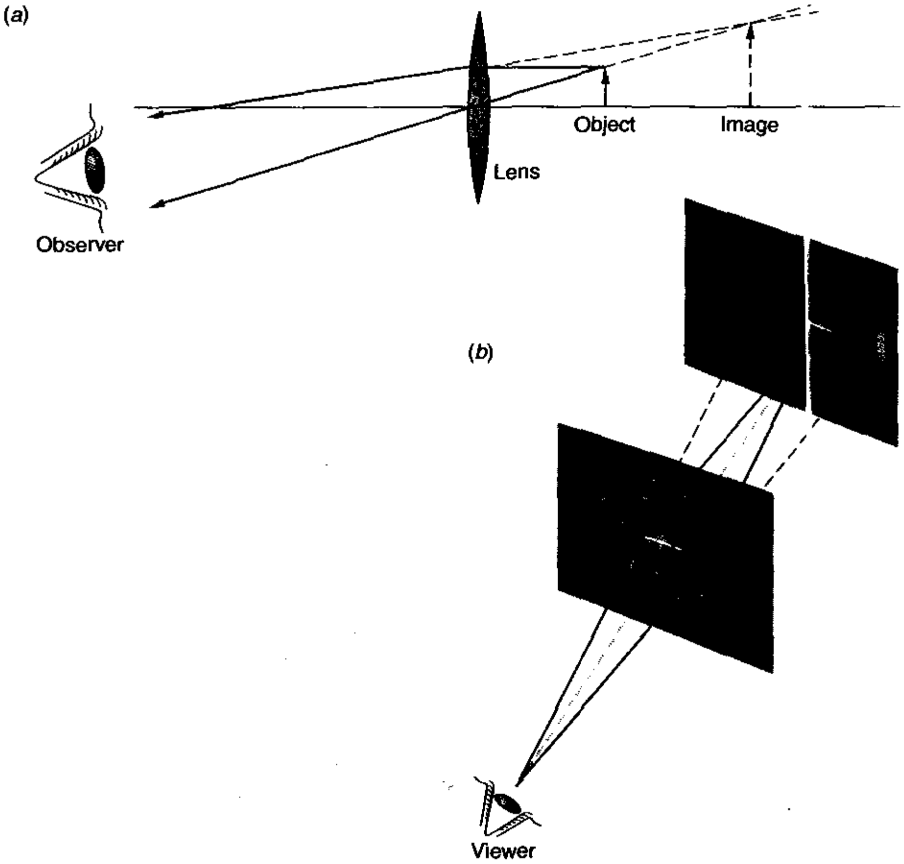
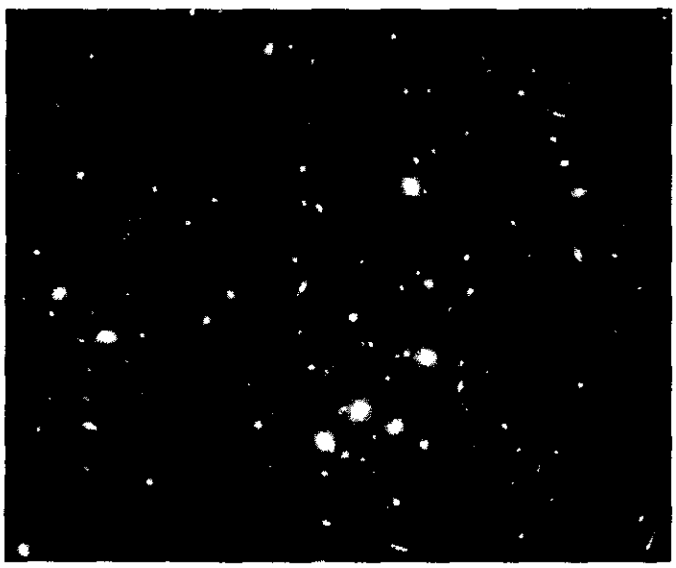


Fig. 2-20 (*a*) Ordinary refracting lens bends light, causing many rays that would not otherwise have reached the observer's eye to do so. Their apparent origin is the image formed by the lens. Notice that the image is not the same size as the object (magnification) and, although not shown here, the shape of the lens can cause the image shape to be different from that of the object. (*b*) Gravitational lens has the same effects on the light from distant galaxies seen at Earth.



Images of distant galaxies are drawn out into arcs by the massive cluster of galaxies Abell 2218, whose enormous gravitational field acts as a lens to magnify, brighten, and distort the images. Abell 2218 is about 2 billion $c \cdot y$ from Earth. The arcs in this January 2000 Hubble Space Telescope photograph are images of galaxies 10 to 20 billion $c \cdot y$ away. [*NASA/ Science VU/ Visuals Unlimited.*]

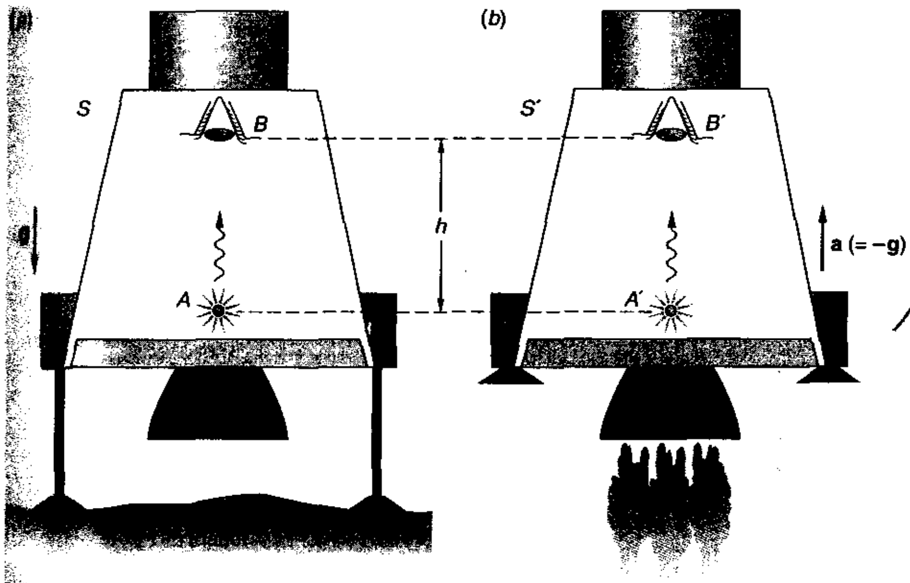


Fig. 2-21 (*a*) System S is at rest in the gravitational field of the planet. (*b*) Spaceship S', far from any mass, accelerates with $\mathbf{a} = -\mathbf{g}$.

divided by c^2. According to the equivalence principle, the detector at B in S must also measure the frequency of the arriving light to be f, even though S is at rest on the planet and, therefore, the shift cannot be due to the Doppler effect! Since the vibrating atom that produced the light pulse at A can be considered to be a clock, and since no "cycles" of the vibration are lost on the pulse's trip from A to B, the observer at B must conclude that the clock at A runs slow, compared with an identical clock (or an identical atom) located at B. Since A is at the lower potential, the observer concludes that clocks run more slowly the lower the gravitational potential. This shift of clock rates to lower frequencies, and hence longer wavelengths, in lower gravitational potentials is the ***gravitational redshift.***

In the more general case of a spherical, nonrotating mass M, the change in gravitational potential between the surface at some distance R from the center and a point at infinity is given by

$$\Delta\phi = \int_R^\infty \frac{GM}{r^2}dr = GM\,(-1/r)\Big|_R^\infty = \frac{GM}{R} \qquad \textbf{2-46}$$

and the factor by which gravity shifts the light frequency is found from

$$\Delta f/f_0 = (f_0 - f)/f_0 = GM/c^2R$$

or

$$f/f_0 = 1 - GM/c^2R \qquad \text{(gravitational redshift)} \qquad \textbf{2-47}$$

Notice that if the light is moving the other way, i.e., from high to low gravitational potential, the limits of integration in Equation 2-46 are reversed and Equation 2-47 becomes

$$f/f_0 = 1 + GM/c^2R \qquad \text{(gravitational blueshift)} \qquad \textbf{2-48}$$

Analyzing the frequency of starlight for gravitational effects is exceptionally difficult because several shifts are present. For example, the light is gravitationally redshifted as it leaves the star and blueshifted as it arrives at Earth. The blueshift near Earth is negligibly small with current measuring technology; however, the Doppler redshift due to the receding of nearby stars and distant galaxies from us as a part of the general expansion of the universe is typically much larger than gravitational effects and, together with thermal frequency broadening in the stellar atmospheres, results in large uncertainties in measurements. Thus, it is quite remarkable that the relativistic prediction of Equation 2-48 has been tested in the relatively small gravitational field of Earth. R. V. Pound and his co-workers,[20] first in 1960 and then again in 1964 with improved precision, measured the shift in the frequency of 14.4-keV gamma rays emitted by ^{57}Fe falling through a height h of only 22.5 m. Using the Mössbauer effect, an extremely sensitive frequency shift measuring technique developed in 1958, their measurements agreed with the predicted fractional blueshift $gh/c^2 = 2.45 \times 10^{-15}$ to within 1 percent. A number of tests of Equations 2-47 and 2-48 have been conducted—using atomic clocks carried in aircraft, as described in Section 1-4; and, in 1980, by R. F. C. Vessot and his co-workers, using a precision microwave transmitter carried to 10,000 km from Earth in a space probe. These, too, agree with the relativistically predicted frequency shift, the latter to one part in 14,000.

QUESTION

7. The frequency f in Equation 2-47 can be shifted to zero by an appropriate value of M/R. What would be the corresponding value of R for a star with the mass of the sun? Speculate on the significance of this result.

More

The inability of Newtonian gravitational theory to account correctly for the observed rate at which the major axis of Mercury's orbit precessed about the sun was a troubling problem, pointing as it did to some subtle failure of the theory. Einstein's first paper on general relativity quantitatively explained the advance of the *Perihelion of Mercury's Orbit,* setting the stage for general relativity to supplant the old Newtonian theory. A clear description of the relativistic explanation is on the home page: www.whfreeman.com/modphysics4e See also Equations 2-49 through 2-51 here, as well as Figure 2-22 and Table 2-2.

More

General relativity includes a gravitational interaction for particles with zero rest mass, such as photons, which are excluded in Newtonian theory. One consequence is the prediction of a *Delay of Light in a Gravitational Field.* This phenomenon and its subsequent observation are described qualitatively on the home page: www.whfreeman.com/modphysics4e See also Equation 2-52 here, as well as Figures 2-23 and 2-24.

Black Holes Black holes were first predicted by Oppenheimer and Snyder in 1939. According to the general theory of relativity, if the density of an object such as a star is great enough, the gravitational attraction will be so large that nothing can escape, not even light or other electromagnetic radiation. It is as if space itself were being drawn inward faster than light could move outward through it. A remarkable property of such an object is that nothing that happens inside it can be communicated to the outside world. This occurs when the gravitational potential at the surface of the mass M becomes so large that the frequency of radiation emitted at the surface is gravitationally redshifted to zero. From Equation 2-47 we see that the frequency will be zero when the radius of the mass has the critical value $R_G = GM/c^2$. This result is a consequence of the principle of equivalence, but Equation 2-47 is a $v \ll c$ approximation. A precise derivation of the critical value of R_G, called the *Schwarzschild radius,* yields

$$R_G = \frac{2GM}{c^2} \qquad \textbf{2-53}$$

For an object of mass equal to that of our sun to be a black hole, its radius would be about 3 km. A large number of possible black holes have been identified by astronomers in recent years, one of them at the center of the Milky Way. (See Chapter 14.)

An interesting historical note is that Equation 2-53 was first derived by the nineteenth-century French physicist Pierre Laplace using Newtonian mechanics to compute the escape velocity v_e from a planet of mass M before anyone had ever heard of Einstein or black holes. The result, derived in first-year physics courses by setting the kinetic energy of the escaping object equal to the gravitational potential at the surface of the planet (or star), is

$$v_e = \sqrt{\frac{2GM}{r}}$$

Setting $v_e = c$ gives Equation 2-53. Laplace obtained the correct result by making two fundamental errors that just happened to cancel one another!

Gravitational Waves Einstein's formulation of general relativity in 1916 explicitly predicted the existence of gravitational radiation. He showed that, just as accelerated electric charges generate time-dependent electromagnetic fields in

Fig. 2-25 Gravitational waves, intense ripples in the fabric of spacetime, are expected to be generated by a merging binary system of neutron stars or black holes. The amplitude decreases with distance due to the *1/R* fall-off and because waves farther from the source were emitted at an earlier time, when the emission was weaker. [*Image courtesy of Caltech/LIGO.*]

space—i.e., electromagnetic waves—accelerated masses would create time-dependent gravitational fields in space—i.e., *gravitational waves* that propagate from their source at the speed of light. The gravitational waves are propagating warpages, or distortions of spacetime. Figure 2-25 illustrates gravitational radiation emitted by two merging black holes distorting the otherwise flat "fabric" of spacetime.

The best experimental evidence that exists thus far in support of the gravitational wave prediction is indirect. In 1974 Hulse and Taylor[24] discovered the first binary pulsar, i.e., a pair of neutron stars orbiting each other, one of which was emitting periodic flashes of electromagnetic radiation (pulses). In an exquisitely precise experiment they showed that the gradual decrease in the orbital period of the pair was in good agreement with the general relativistic prediction for the rate of loss of gravitational energy via the emission of gravitational waves.

This application of Michelson's interferometer may well lead to the first direct detection of "ripples," or waves in spacetime.

Experiments are currently under way in several countries to detect gravitational waves arriving at Earth *directly*. One of the most promising is LIGO (*L*aser *I*nterferometer *G*ravitational-Wave *O*bservatory), a pair of large Michelson interferometers, one in Louisiana and the other 3030 km away in Washington, operating in coincidence. Figure 2-26 illustrates one of the LIGO interferometers. Each arm is 4 km long. The laser beams are reflected back and forth by the mirrors dozens of times before recombining at the photodetector, making the effective lengths of the arms about 400 km. The arrival of a gravitational wave would stretch one arm of the

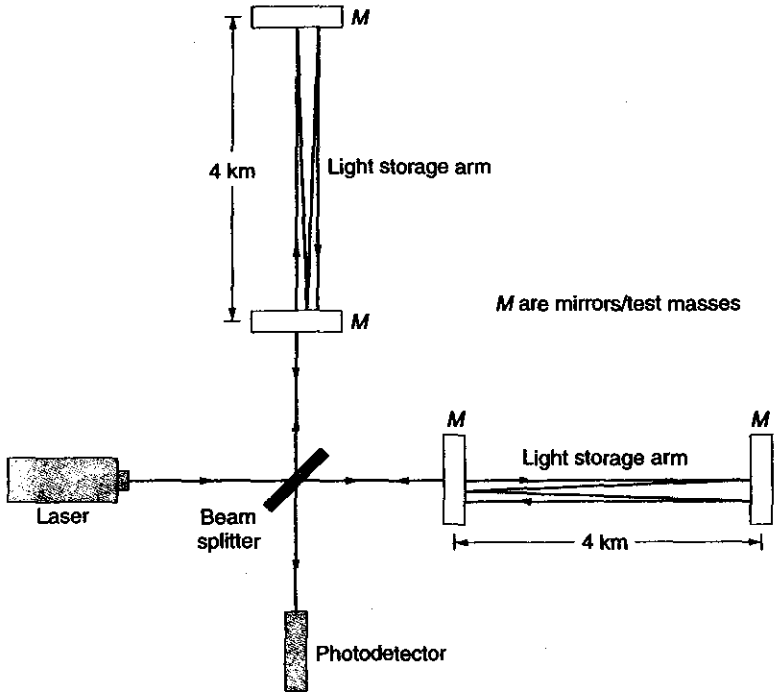


Fig. 2-26 The LIGO detectors are equal-arm Michelson interferometers. The mirrors, each 25 cm in diameter by 10 cm thick and isolated from Earth's motions, are also the test masses of the gravitational wave detector. Arrival of a gravitational wave would change the length of each arm by about the diameter of an atomic nucleus and result in interference fringes at the photodetector.

interferometer by about 1/1000th of the diameter of an atom and squeeze the other arm by the same minuscule amount! Nonetheless, that tiny change in the lengths is sufficient to put the recombining laser beams slightly out of phase and produce interference fringes. The two LIGO interferometers must record the event within 10 ms of each other for the signal to be interpreted as a gravitational wave. LIGO completed its two-year, low-sensitivity initial operational phase and went online in mid-2002. None of the half-dozen experiments under way around the world has yet confirmed detection of a gravitational wave.[25]

There is still an enormous amount to be learned about the predictions and implications of general relativity—not just about such things as black holes and gravity waves, but also, for example, about gravity and spacetime in the very early universe, when forces were unified and the constituents were closely packed. These and other fascinating matters are investigated more specifically in the areas of astrophysics and cosmology (Chapter 14) and particle physics (Chapter 13), fields linked together by general relativity, perhaps the grandest of Einstein's great scientific achievements.

QUESTION

8. Speculate on what the two errors made by Laplace in deriving Equation 2-53 might have been.

Summary

TOPIC	RELEVANT EQUATIONS AND REMARKS	
1. Relativistic momentum	$\mathbf{p} = \gamma m\mathbf{u}$	2-7
	The relativistic momentum is conserved and approaches $m\mathbf{u}$ for $v \ll c$. $\gamma = (1 - u^2/c^2)^{-1/2}$ in Equation 2-7, where u = particle speed in S.	
2. Relativistic energy	$E = \gamma mc^2$	2-10
Total energy	The relativistic total energy is conserved.	
Kinetic energy	$E_k = \gamma mc^2 - mc^2$	2-9
	The rest energy is mc^2. $\gamma = (1 - u^2/c^2)^{-1/2}$ in Equations 2-9 and 2-10.	
3. Lorentz transformation for E and $\mathbf{p}$	$p_x' = \gamma(p_x - vE/c^2)$ $\quad p_y' = p_y$ $E' = \gamma(E - vp_x)$ $\quad p_z' = p_z$	2-16
	where v = relative speed of the systems and $\gamma = (1 - v^2/c^2)^{-1/2}$	
4. Mass/energy conversion	Whenever additional energy ΔE in any form is stored in an object, the rest mass of the object is increased by $\Delta m = \Delta E/c^2$.	
5. Invariant mass	$(mc^2)^2 = E^2 - (pc)^2$	2-32
	The energy and momentum of any system combine to form an invariant four-vector whose magnitude is the rest energy of the mass m.	
6. Force in relativity	The force $\mathbf{F} = m\mathbf{a}$ is not invariant in relativity. Relativistic force is defined as $\mathbf{F} = \dfrac{d\mathbf{p}}{dt} = \dfrac{d(\gamma m\mathbf{u})}{dt}$	2-8
7. General relativity principle of equivalence	A homogeneous gravitational field is completely equivalent to a uniformly accelerated frame.	

General References

The following general references are written at a level appropriate for the readers of this book.

Bohm, D., *The Special Theory of Relativity,* W. A. Benjamin, New York, 1965.

French, A. P., *Albert Einstein: A Centenary Volume,* Harvard University Press, Cambridge, Mass., 1979. This is an excellent collection of contributions from many people about Einstein's life and work.

French, A. P., *Special Relativity,* Norton, New York, 1968.

Lorentz, H. A., A. Einstein, H. Minkowski, and W. Weyl, *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity* (trans. W. Perrett and J. B. Jeffery), Dover, New York, 1923. Two of Einstein's papers reprinted here are of interest in connection with this chapter: "On the Electrodynamics of Moving Bodies," [*Annalen der Physik,* **17** (1905)], and "Does the Inertia of a Body Depend upon Its Energy Content?" [*Annalen der Physik,* **17** (1905)].

Pais, A., *Subtle Is the Lord . . . ,* Oxford University Press, Oxford, 1982.

Resnick, R., *Introduction to Relativity,* Wiley, New York, 1968.

Resnick, R., and D. Halliday, *Basic Concepts in Relativity and Early Quantum Theory,* 2d ed., Macmillan, New York, 1992.

"Resource Letter SRT-1 on Special Relativity Theory," *American Journal of Physics,* **30**, 462 (1962). This is a list of references.

Rosser, W. G. V., *The Theory of Relativity*, Butterworth, London, 1964.

Special Relativity Theory: Selected Reprints, American Association of Physics Teachers, New York, 1963. Booklet containing some of the papers listed in "Resource Letter SRT-1."

Taylor, E. F., and J. A. Wheeler, *Spacetime Physics*, 2d ed., W. H. Freeman & Co., 1992. A good book with many examples, problems, and diagrams.

NOTES

1. This *gedankenexperiment* ("thought experiment") is based on one first suggested by G. N. Lewis and R. C. Tolman, *Philosophical Magazine*, **18**, 510 (1909).
2. You can see that this is so by rotating Figure 2-1*a* through 180° on its own plane; it then matches Figure 2-1*b* exactly.
3. C. G. Adler, *American Journal of Physics*, **55**, 739 (1987).
4. This idea grew out of the results of the measurements of masses in chemical reactions in the nineteenth century, which, within the limits of experimental uncertainties of the time, were always observed to conserve mass. The conservation of energy had a similar origin in the experiments of James Joule (1818–1889) as interpreted by Hermann von Helmholtz (1821–1894). This is not an unusual way for conservation laws to originate; they still do it this way.
5. The approximation of Equation 2-10 used in this discussion was, of course, not developed from Newton's equations. The rest energy mc^2 has no classical counterpart.
6. "Facilitates" means that we don't have to make frequent unit conversions or carry along large powers of 10 with nearly every factor in many calculations. However, a word of caution is in order: Always remember that the eV is *not* a basic SI unit. When making calculations whose results are to be in SI units, don't forget to convert the eV!
7. A. Einstein, *Annalen der Physik*, **17**, 1905.
8. Strictly speaking, the time component should be written $ic\,\Delta t$, where $i = (-1)^{1/2}$. The i is the origin of the minus sign in the spacetime interval, as well as in Equation 2-32 for the energy/momentum four-vector and other four-vectors in both special and general relativity. Its inclusion was a contribution of Hermann Minkowski (1864–1909), a Russian-German mathematician, who developed the geometric interpretation of relativity and who was one of Einstein's professors in Zurich. Consideration of the four-dimensional geometry is beyond the scope of our discussions, so we will not be concerned with the i.
9. Other conservation laws of physics must also be satisfied, e.g., electric charge, angular momentum.
10. The positron is a particle with the same mass as an ordinary electron, but with a positive electric charge of the same magnitude as that carried by the electron. It and other antiparticles will be discussed in Chapters 11 and 13.
11. Since electrons are thought to be point particles, i.e., they have no space dimensions, it isn't clear what it means to "hit" an electron. Think of it as the photon coming close to the electron's location, hence, in its strong local electric field.
12. Such a system is called a *polyelectron*. It is analogous to an ionized hydrogen molecule, much as positronium is analogous to a hydrogen atom. (See caption for Figure 2-12.)
13. Satellite navigation systems, e.g., the U.S. Air Force Global Positioning System, are now so precise that the minute corrections arising primarily from the general relativistic time dilation must be taken into account by the systems' programs.
14. From Einstein's lecture in Kyoto in late 1922. See A. Pais, *Subtle Is the Lord . . .* (Oxford: Oxford University Press, 1982).
15. From an unpublished paper now in the collection of the Pierpont Morgan Library in New York. See Pais, *Subtle Is the Lord . . .* (Oxford: Oxford University Press, 1982).
16. Einstein inquired of the astronomer George Hale (after whom the 5-m telescope on Palomar is named) in 1913 whether such minute deflections could be measured near the sun. The answer was no, but a corrected calculation two years later doubled the predicted deflection and brought detection to within the realm of possibility.
17. This is not a simple integration. See, e.g., Adler et al., *Introduction to General Relativity* (New York: McGraw-Hill, 1965).
18. Both Newtonian mechanics and special relativity predict half of this value. The particle-scattering formula used in Chapter 4 to obtain Equation 4-3 applied to the gravitational deflection of a photon of mass hv/c^2 by the solar mass $M_\odot$ at impact parameter b equal to the solar radius $R_\odot$ shows how this value arises.
19. A copy of Einstein's work (he was then in Berlin) was smuggled out of Germany to Eddington in England so that he could plan the project. Germany and England were then at war. Arthur S. Eddington (1882–1944) was then director of the prestigious Cambridge Observatory. British authorities approved the eclipse expeditions in order to avoid the embarrassment of putting such a distinguished scientist as Eddington, a conscientious objector, into a wartime internment camp.
20. See, for example, R. V. Pound and G. A. Rebka, Jr., *Physical Review Letters*, **4**, 337 (1960).
21. These values are relative to the fixed stars.
22. A. Einstein, "The Foundation of the General Theory of Relativity," *Annalen der Physik*, **49**, 769 (1916).

23. I. I. Shapiro et al., *Physical Review Letters*, **26**, 1132 (1971).
24. R. A. Hulse and J. H. Taylor, *Astrophysical Journal*, **195**, L51 (1975).
25. Gravity wave detectors outside the United States are the TAMA 300 (Japan), GEO 600 (Germany), and Virgo (Italy). NASA and the European Space Agency are designing a space-based gravity wave detector, LISA, that will have arms 5 million kilometers long. The three satellites that LISA will comprise are scheduled for launch in 2011.

PROBLEMS

Level I

Section 2-1 Relativistic Momentum and Section 2-2 Relativistic Energy

2-1. Show that $p_{yA} = -p_{yB}$, where p_{yA} and p_{yB} are the relativistic momenta of the balls on Figure 2-1, given by

$$p_{yA} = \frac{mu_0}{\sqrt{1 - u_0^2/c^2}} \qquad p_{yB} = \frac{mu_{yB}}{\sqrt{1 - (u_{xB}^2 + u_{yB}^2)/c^2}}$$

$$u_{yB} = -u_0\sqrt{1 - \frac{v^2}{c^2}} \qquad u_{xB} = v$$

2-2. Show that $d(\gamma mu) = m(1 - u^2/c^2)^{-3/2}\, du$.

2-3. An electron of rest energy $mc^2 = 0.511$ MeV moves with respect to the laboratory at speed $u = 0.6\,c$. Find (*a*) γ, (*b*) p in units of MeV/c, (*c*) E, and (*d*) E_k.

2-4. How much energy would be required to accelerate a particle of mass m from rest to a speed of (*a*) 0.5 c, (*b*) 0.9 c, and (*c*) 0.99 c? Express your answers as multiples of the rest energy.

2-5. Two 1-kg masses are separated by a spring of negligible mass. They are pushed together, compressing the spring. If the work done in compressing the spring is 10 J, find the change in mass of the system in kilograms. Does the mass increase or decrease?

2-6. At what value of u/c does the measured mass of a particle exceed its rest mass by (*a*) 10%, (*b*) a factor of 5, and (*c*) a factor of 20?

2-7. A cosmic ray proton is moving at such a speed that it can travel from the moon to Earth in 1.5 s. (*a*) At what fraction of the speed of light is the proton moving? (*b*) What is its kinetic energy? (*c*) What value would be measured for its mass by an observer in the Earth reference frame? (*d*) What percent error is made in the kinetic energy by using the classical relation? (The Earth-moon distance is 3.8×10^5 km. Ignore Earth's rotation.)

2-8. How much work must be done on a proton to increase its speed from (*a*) $0.15c$ to $0.16c$? (*b*) $0.85c$ to $0.86c$? (*c*) $0.95c$ to $0.96c$? Notice that the change in the speed is the same in each case.

2-9. The Relativistic Heavy Ion Collider (RHIC) at Brookhaven is colliding fully ionized gold (Au) nuclei accelerated to an energy of 200 GeV per nucleon. Each Au nucleus contains 197 nucleons. (*a*) What is the speed of each Au nucleus just before collision? (*b*) What is the momentum of each at that instant? (*c*) What energy and momentum would be measured for one of the Au nuclei by an observer in the rest system of the other Au nucleus?

2-10. (*a*) Compute the rest energy of 1 g of dirt. (*b*) If you could convert this energy entirely into electrical energy and sell it for 10 cents per kilowatt-hour, how much money would you get? (*c*) If you could power a 100-W light bulb with the energy, for how long could you keep the bulb lit?

2-11. An electron with rest energy of 0.511 MeV moves with speed $u = 0.2c$. Find its total energy, kinetic energy, and momentum.

2-12. A proton with rest energy of 938 MeV has a total energy of 1400 MeV. (*a*) What is its speed? (*b*) What is its momentum?
2-13. The total energy of a particle is twice its rest energy. (*a*) Find u/c for the particle. (*b*) Show that its momentum is given by $p = (3)^{1/2}\, mc$.
2-14. An electron in a hydrogen atom has a speed about the proton of 2.2×10^6 m/s. (*a*) By what percent do the relativistic and Newtonian values of E_k differ? (*b*) By what percent do the momentum values differ?
2-15. Suppose that you seal an ordinary 60-W light bulb and a suitable battery inside a transparent enclosure and suspend the system from a very sensitive balance. (*a*) Compute the change in the mass of the system if the lamp is on continuously for one year at full power. (*b*) What difference, if any, would it make if the inner surface of the container were a perfect reflector?

Section 2-3 Mass/Energy Conversion and Binding Energy

2-16. Use Appendix A and Table 2-1 to find how much energy is needed to remove one proton from a ^{4}He atom, leaving a ^{3}H atom plus a proton and an electron.
2-17. Use Appendix A and Table 2-1 to find how much energy is required to remove one of the neutrons from a ^{3}H atom to yield a ^{2}H atom plus a neutron?
2-18. The energy released when sodium and chlorine combine to form NaCl is 4.2 eV. (*a*) What is the increase in mass (in unified mass units) when a molecule of NaCl is dissociated into an atom of Na and an atom of Cl? (*b*) What percentage error is made in neglecting this mass difference? (The mass of Na is about 23 u, and that of Cl is about 35.5 u.)
2-19. In a nuclear fusion reaction two ^{2}H atoms are combined to produce ^{4}He. (*a*) Calculate the decrease in rest mass in unified mass units. (*b*) How much energy is released in this reaction? (*c*) How many such reactions must take place per second to produce 1 W of power?
2-20. Calculate the rate of conversion of rest mass to energy (in kg/h) needed to produce 100 MW.
2-21. When a beam of high-energy protons collides with protons at rest in the laboratory (e.g., in a container of water or liquid hydrogen), neutral pions (π^0) are produced by the reaction $p + p \rightarrow p + p + \pi^0$. Compute the threshold energy of the protons in the beam for this reaction to occur. (See Table 2-1 and Example 2-11.)
2-22. The energy released in the fission of a ^{235}U nucleus is about 200 MeV. How much rest mass (in kg) is converted to energy in this fission?

Section 2-4 Invariant Mass

2-23. The K^0 particle decays according to the equation $K^0 \rightarrow \pi^+ + \pi^-$. If a particular K^0 decays while it is at rest in the laboratory, what are the kinetic energies of each of the two pions? (The rest mass of the K^0 is 497.7 MeV/c^2.)
2-24. Compute the force exerted on the palm of your hand by the beam from a 1.0-W flashlight (*a*) if your hand absorbs the light, and (*b*) if the light reflects from your hand. What would be the mass of a particle that exerts that same force in each case if you hold it at Earth's surface?
2-25. An electron-positron pair combined as positronium is at rest in the laboratory. The pair annihilate, producing a pair of photons (gamma rays) moving in opposite directions in the lab. Show that the invariant rest energy of the gamma rays is equal to that of the electron pair.
2-26. Show that Equation 2-31 can be written $E = mc^2(1 + p^2/m^2c^2)^{1/2}$ and use the binomial expansion to show that, when pc is much less than mc^2, $E \approx mc^2 + p^2/2m$.
2-27. An electron of rest energy 0.511 MeV has a total energy of 5 MeV. (*a*) Find its momentum in units of MeV/c. (*b*) Find u/c.
2-28. Make a sketch of the total energy of an electron E as a function of its momentum p. (See Equations 2-36 and 2-41 for the behavior of E at large and small values of p.)

2-29. What is the speed of a particle that is observed to have momentum 500 MeV/c and energy 1746 MeV. What is the particle's mass (in MeV/c^2)?
2-30. An electron of total energy 4.0 MeV moves perpendicular to a uniform magnetic field along a circular path whose radius is 4.2 cm. (*a*) What is the strength of the magnetic field **B**? (*b*) By what factor does γm exceed m?
2-31. A proton is bent into a circular path of radius 2 m by a magnetic field of 0.5 T. (*a*) What is the momentum of the proton? (*b*) What is its kinetic energy?

Section 2-5 General Relativity

2-32. Compute the deflection angle α for light from a distant star that would, according to general relativity, be measured by an observer on the moon as the light grazes the edge of Earth.
2-33. A set of twins work in the Sears Tower, a very tall office building in Chicago. One works on the top floor and the other works in the basement. Considering general relativity, which twin will age more slowly? (*a*) They will age at the same rate. (*b*) The twin who works on the top floor will age more slowly. (*c*) The twin who works in the basement will age more slowly. (*d*) It depends on the building's speed. (*e*) None of the previous choices is correct.
2-34. Jupiter makes 8.43 orbits/century and exhibits an orbital eccentricity $\epsilon = 0.048$. Jupiter is 5.2 AU from the sun and has a mass 318 times the Earth's 5.98×10^{24} kg. What does general relativity predict for the rate of precession of Jupiter's perihelion? (It has not yet been measured.) (The astronomical unit AU = the mean Earth-sun distance $= 1.50 \times 10^{11}$ m.)
2-35. A synchronous satellite "parked" in orbit over the equator is used to relay microwave transmissions between stations on the ground. To what frequency must the satellite's receiver be tuned if the frequency of the transmission from Earth is exactly 9.375 GHz? (Ignore all Doppler effects.)
2-36. A particular distant star is found to be 92 $c \cdot$y from Earth. On a direct line between us and the star and 35 $c \cdot$y from the distant star is a dense white dwarf star with a mass equal to 3 times the sun's mass $M_\odot$ and a radius of 10^4 km. Deflection of the light beam from the distant star by the white dwarf causes us to see it as a pair of circular arcs like those shown in Figure 2-20(*b*). Find the angle 2α formed by the lines of sight to the two arcs.

Level II

2-37. A clock is placed on a satellite that orbits Earth with a period of 90 min at an altitude of 300 km. By what time interval will this clock differ from an identical clock on Earth after 1 year? (Include both special and general relativistic effects.)
2-38. Referring to Example 2-11, find the total energy E' as measured in S' where $\mathbf{p}' = 0$.
2-39. In the Stanford linear collider, small bundles of electrons and positrons are fired at each other. In the laboratory's frame of reference, each bundle is about 1 cm long and 10 μm in diameter. In the collision region, each particle has an energy of 50 GeV, and the electrons and positrons are moving in opposite directions. (*a*) How long and how wide is each bundle in its own reference frame? (*b*) What must be the minimum proper length of the accelerator for a bundle to have both its ends simultaneously in the accelerator in its own reference frame? (The actual length of the accelerator is less than 1000 m.) (*c*) What is the length of a positron bundle in the reference frame of the electron bundle? (*d*) What are the momentum and energy of the electrons in the rest frame of the positrons?
2-40. The rest energy of a proton is about 938 MeV. If its kinetic energy is also 938 MeV, find (*a*) its momentum and (*b*) its speed.
2-41. A spaceship of mass 10^6 kg is coasting through space when suddenly it becomes necessary to accelerate. The ship ejects 10^3 kg of fuel in a very short time at a speed of

$c/2$ relative to the ship. (*a*) Neglecting any change in the rest mass of the system, calculate the speed of the ship in the frame in which it was initially at rest. (*b*) Calculate the speed of the ship using classical Newtonian mechanics. (*c*) Use your results from (*a*) to estimate the change in the rest mass of the system.

2-42. Professor Spenditt, oblivious to economics and politics, proposes the construction of a circular proton accelerator around Earth's circumference using bending magnets that provide a magnetic field of 1.5 T. (*a*) What would be the kinetic energy of protons orbiting in this field in a circle of radius R_E? (*b*) What would be the period of rotation of these protons?

2-43. In ancient Egypt the annual flood of the Nile was predicted by the rise of Sirius (the Dog Star). Sirius is one of a binary pair whose companion is a white dwarf. Orbital analysis of the pair indicates that the dwarf's mass is 2×10^{30} kg (i.e., about one solar mass). Comparison of spectral lines emitted by the white dwarf with those emitted by the same element on Earth shows a fractional frequency shift of 7×10^{-4}. Assuming this to be due to a gravitational redshift, compute the density of the white dwarf. (For comparison, the sun's density is 1409 kg/m^3.)

2-44. Show that the creation of an electron-position pair (or any particle-antiparticle pair, for that matter) by a single photon is not possible in isolation, i.e., that additional mass (or radiation) must be present. (*Hint:* Use the conservation laws.)

2-45. With inertial systems S and S' arranged with their corresponding axes parallel and S' moving in the $+x$ direction, it was apparent that the Lorentz transformation for y and z would be $y' = y$ and $z' = z$. The transformation for the y and z components of the momentum are not so apparent, however. Show that, as stated in Equations 2-16 and 2-17, $p'_y = p_y$ and $p'_z = p_z$. p. 79

Level III

2-46. Two identical particles of rest mass m are each moving toward the other with speed u in frame S. The particles collide inelastically with a spring that locks shut (Figure 2-9) and come to rest in S, and their initial kinetic energy is transformed into potential energy. In this problem you are going to show that the conservation of momentum in reference frame S', in which one of the particles is initially at rest, requires that the total rest mass of the system after the collision be $2m/(1 - u^2/c^2)^{1/2}$. (*a*) Show that the speed of the particle not at rest in frame S' is

$$u' = \frac{2u}{1 + u^2/c^2}$$

and use this result to show that

$$\sqrt{1 - \frac{u'^2}{c^2}} = \frac{1 - u^2/c^2}{1 + u^2/c^2}$$

(*b*) Show that the initial momentum in frame S' is $p' = 2mu/(1 - u^2/c^2)$. (*c*) After the collision, the composite particle moves with speed u in S' (since it is at rest in S). Write the total momentum after the collision in terms of the final rest mass M, and show that the conservation of momentum implies that $M = 2m/(1 - u^2/c^2)^{1/2}$. (*d*) Show that the total energy is conserved in each reference frame.

2-47. An antiproton $\bar{p}$ has the same rest energy as a proton. It is created in the reaction $p + p \rightarrow p + p + p + \bar{p}$. In an experiment, protons at rest in the laboratory are bombarded with protons of kinetic energy E_k, which must be great enough so that kinetic energy equal to $2mc^2$ can be converted into the rest energy of the two particles. In the frame of the laboratory, the total kinetic energy cannot be converted into rest energy because of conservation of momentum. However, in the zero-momentum reference frame in which the two initial protons are moving toward each other with equal speed u, the total kinetic energy can be converted into rest energy. (*a*) Find the speed of each proton u such that the total kinetic energy in the zero-momentum frame is $2mc^2$. (*b*) Transform to the laboratory's frame in which one proton is at rest, and find the speed u' of the other proton. (*c*) Show that the kinetic energy of the moving proton in the laboratory's frame is $E_k = 6mc^2$.

2-48. In a simple thought experiment, Einstein showed that there is mass associated with electromagnetic radiation. Consider a box of length L and mass M resting on a frictionless surface. At the left wall of the box is a light source that emits radiation of energy E, which is absorbed at the right wall of the box. According to classical electromagnetic theory, this radiation carries momentum of magnitude $p = E/c$. (*a*) Find the recoil velocity of the box such that momentum is conserved when the light is emitted. (Since p is small and M is large, you may use classical mechanics.) (*b*) When the light is absorbed at the right wall of the box, the box stops, so the total momentum remains zero. If we neglect the very small velocity of the box, the time it takes for the radiation to travel across the box is $\Delta t = L/c$. Find the distance moved by the box in this time. (*c*) Show that if the center of mass of the system is to remain at the same place, the radiation must carry mass $m = E/c^2$.

2-49. A pion spontaneously decays into a muon and an antineutrino according to (among other processes) $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. Current experimental evidence indicates that the mass m of the $\bar{\nu}_\mu$ is no greater that about 190 keV and may, in fact, be zero. Assuming that the pion decays at rest in the laboratory, compute the energies and momenta of the muon and muon antineutrino (*a*) if the mass of the antineutrino is zero and (*b*) if its mass is 190 keV. The mass of the pion is 139.56755 MeV/c^2 and the mass of the muon is 105.65839 MeV/c^2.

2-50. Use Equation 2-47 to obtain the gravitational redshift in terms of the wavelength λ. Use that result to determine the shift in wavelength of light emitted by a white dwarf star at 720.00 nm. Assume the white dwarf has the same mass as the sun (1.99×10^{30} kg), but a radius equal to only 1 percent of the solar radius $R_\odot$. ($R_\odot = 6.96 \times 10^8$ m.)

2-51. For a particle moving in the xy plane of S, show that the y' component of the acceleration is given by

$$a'_y = \frac{a_y}{\gamma^2(1 - u_x v/c^2)} + \frac{a_x u_y v/c^2}{\gamma^2(1 - u_x v/c^2)^3}$$

2-52. Consider an object of mass m at rest in S acted upon by a force **F** with components F_x and F_y. System S' moves with instantaneous velocity **v** in the $+x$ direction. Defining the force with Equation 2-8 and using the Lorentz velocity transformation, show that (*a*) $F'_x = F_x$ and (*b*) $F'_y = F_y/\gamma$. (*Hint*: See Problem 2-51.)

2-53. An unstable particle of mass M decays into two identical particles, each of mass m. Obtain an expression for the velocities of the two decay particles in the lab frame (*a*) if M is at rest in the lab and (*b*) if M has total energy $4mc^2$ when it decays and the decay particles move along the direction of M.