Element	P	Ca	Co	Kr	Mo	I
Z	15	20	27	36	42	53
Wavelength (nm)	10.41	4.05	1.79	0.73	0.51	0.33

4-52. In this problem you are to obtain the Bohr results for the energy levels in hydrogen without using the quantization condition of Equation 4-17. In order to relate Equation 4-14 to the Balmer-Ritz formula, assume that the radii of allowed orbits are given by $r_n = n^2 r_0$, where n is an integer and r_0 is a constant to be determined. (*a*) Show that the frequency of radiation for a transition to $n_f = n - 1$ is given by $f \approx kZe^2/hr_0n^3$ for large n. (*b*) Show that the frequency of revolution is given by

$$f^2_{rev} = \frac{kZe^2}{4\pi^2 m\, r_0^3 n^6}$$

(*c*) Use the correspondence principle to determine r_0 and compare with Equation 4-19.
4-53. Calculate the energies and speeds of electrons in circular Bohr orbits in a hydrogenlike atom using the relativistic expressions for kinetic energy and momentum.
4-54. (*a*) Write a computer program for your personal computer or programmable calculator that will provide you with the spectral series of H-like atoms. Inputs to be included are n_i, n_f, Z, and the nuclear mass M. Outputs are to be the wavelengths and frequencies of the first six lines and the series limit for the specified n_f, Z, and M. Include the reduced mass correction. (*b*) Use the program to compute the wavelengths and frequencies of the Balmer series. (*c*) Pick an $n_f > 100$, name the series the [your name] series, and use your program to compute the wavelengths and frequencies of the first three lines and the limit.
4-55. Figure 4-25 shows an energy-loss spectrum for He measured in an apparatus such as that shown in Figure 4-23*a*. Use the spectrum to construct and draw carefully to scale an energy-level diagram for He.

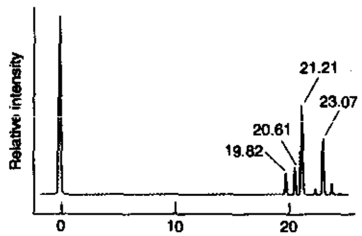

Fig. 4-25 Energy-loss spectrum of helium. Incident electron energy was 34 eV. The elastically scattered electrons cause the peak at 0 eV.

4-56. If electric charge did not exist and electrons were bound to protons by the gravitational force to form hydrogen, derive the corresponding expressions for a_0 and E_n and compute the energy and frequency of the H_α line and the limit of the Balmer series. Compare these with the corresponding quantities for "real" hydrogen.
4-57. A sample of hydrogen atoms are all in the $n = 5$ state. If all the atoms return to the ground state, how many different photon energies will be emitted, assuming all possible transitions occur? If there are 500 atoms in the sample and assuming that from any state all possible downward transitions are equally probable, what is the total number of photons that will be emitted when all of the atoms have returned to the ground state?

Chapter 5 The Wavelike Properties of Particles

In 1924, a French graduate student, Louis de Broglie,[1] proposed in his doctoral dissertation that the dual—i.e., wave-particle—behavior that was by then known to exist for radiation was also a characteristic of matter, in particular, electrons. This suggestion was highly speculative, since there was yet no experimental evidence whatsoever for any wave aspects of electrons or any other particles. What had led him to this seemingly strange idea? It was a "bolt out of the blue," like Einstein's "happy thought," that led to the principle of equivalence (see Chapter 2). De Broglie described it with these words:

> After the end of World War I, I gave a great deal of thought to the theory of quanta and to the wave-particle dualism. . . . It was then that I had a sudden inspiration. Einstein's wave-particle dualism was an absolutely general phenomenon extending to all physical nature.[2]

Since the visible universe consists entirely of matter and radiation, de Broglie's hypothesis is a fundamental statement about the grand symmetry of nature. (There is currently strong observational evidence that approximately 70 percent of the universe consists of some sort of invisible "dark energy." See Chapter 14.)

5-1 The de Broglie Hypothesis

De Broglie stated his proposal mathematically with the following equations for the frequency and wavelength of the electron waves, which are referred to as the *de Broglie relations:*

$$f = \frac{E}{h} \qquad 5\text{-}1$$

$$\lambda = \frac{h}{p} \qquad 5\text{-}2$$

where E is the total energy, p is the momentum, and λ is called the *de Broglie wavelength* of the particle. For photons, these same equations result directly from

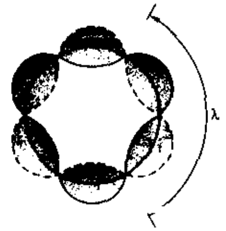

Fig. 5-1 Standing waves around the circumference of a circle. In this case the circle is 3λ in circumference. If the vibrator were, for example, a steel ring that had been suitably tapped with a hammer, the shape of the ring would oscillate between the extreme positions represented by the solid and broken lines.

Einstein's quantization of radiation $E = hf$ and Equation 2-31 for a particle of zero rest energy $E = pc$ as follows:

$$E = pc = hf = \frac{hc}{\lambda}$$

By a more indirect approach using relativistic mechanics, de Broglie was able to demonstrate that Equations 5-1 and 5-2 also apply to particles with mass. He then pointed out that these equations lead to a physical interpretation of Bohr's quantization of the angular momentum of the electron in hydrogenlike atoms, namely, that the quantization is equivalent to a standing-wave condition (see Figure 5-1). We have

$$mvr = n\hbar = \frac{nh}{2\pi} \qquad \text{for} \qquad n = \text{integer}$$

$$2\pi r = \frac{nh}{mv} = \frac{nh}{p} = n\lambda = \text{circumference of orbit} \qquad \textbf{5-3}$$

The idea of explaining discrete energy states in matter by standing waves thus seemed quite promising.

De Broglie's ideas were expanded and developed into a complete theory by Erwin Schrödinger late in 1925. In 1927, C. J. Davisson and L. H. Germer verified the de Broglie hypothesis directly by observing interference patterns, a characteristic of waves, with electron beams. We will discuss both Schrödinger's theory and the Davisson-Germer experiment in later sections, but first we have to ask ourselves why wavelike behavior of matter had not been observed before de Broglie's work. We can see why if we first recall that the wave properties of light were not noticed, either, until apertures or slits with dimensions of the order of the wavelength of light could be obtained. This is because the wave nature of light is not evident in experiments where the primary dimensions of the apparatus are large compared with the wavelength of the light used. For example, if A represents the diameter of a lens or the width of a slit, then diffraction effects[3] (a manifestation of wave properties) are limited to angles θ around the forward direction ($\theta = 0°$) where $\sin\theta = \lambda/A$. In geometric (ray) optics $\lambda/A \to 0$, so $\theta \approx \sin\theta \to 0$, too. However, if a characteristic dimension of the apparatus becomes of the order of (or smaller than) λ, the wavelength of light passing through the system, then $\lambda/A \to 1$. In that event $\theta \approx \lambda/A$ is readily observable, and the wavelike properties of light become apparent. Because Planck's constant is so small, the wavelength given by Equation 5-2 is extremely small for any macroscopic object. This point is among those illustrated in the following section.

5-2 Measurements of Particle Wavelengths

Although we now have diffraction systems of nuclear dimensions, the smallest-scale systems to which de Broglie's contemporaries had access were the spacings between the planes of atoms in crystalline solids, about 0.1 nm. This means that even for an extremely small macroscopic particle, such as a grain of dust ($m \approx 0.1$ mg) moving through air with the average kinetic energy of the atmospheric gas molecules, the smallest diffraction systems available would have resulted in diffraction angles θ

Louis V. de Broglie, who first suggested that electrons might have wave properties. [*Courtesy of Culver Pictures.*]

only of the order of 10^{-10} radians, far below the limit of experimental detectability. The small magnitude of Planck's constant ensures that λ will be smaller than any readily accessible aperture, placing diffraction beyond the limits of experimental observation. For objects whose momenta are larger than that of the dust particle, the possibility of observing *particle* or *matter waves* is even less, as the following example illustrates.

EXAMPLE 5-1 **De Broglie Wavelength of a Ping-Pong Ball** What is the de Broglie wavelength of a Ping-Pong ball of mass 2.0 g after it is slammed across the table with a speed of 5 m/s?

Solution

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{(2.0 \times 10^{-3}\,\text{kg})(5\,\text{m/s})}$$
$$= 6.6 \times 10^{-32}\,\text{m} = 6.6 \times 10^{-23}\,\text{nm}$$

This is 17 orders of magnitude smaller than typical nuclear dimensions, far below the dimensions of any possible aperture.

The case is different for low-energy electrons, as de Broglie himself realized. At his *soutenance de thèse* (defense of the thesis), de Broglie was asked by Perrin[4] how his hypothesis could be verified, to which he replied that perhaps passing particles, such as electrons, through very small slits would reveal the waves. Consider an electron that has been accelerated through V_0 volts. Its kinetic energy (nonrelativistic) is then

$$E = \frac{p^2}{2m} = eV_0$$

Solving for p and substituting into Equation 5-2,

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{(2mc^2 eV_0)^{1/2}}$$

Using $hc = 1.24 \times 10^3$ eV·nm and $mc^2 = 0.511 \times 10^6$ eV, we obtain

$$\lambda = \frac{1.226}{V_0^{1/2}} \text{ nm} \quad \text{for} \quad eV_0 \ll mc^2 \qquad 5\text{-}4$$

The following example computes an electron de Broglie wavelength, giving a measure of just how small the slit must be.

EXAMPLE 5-2 **De Broglie Wavelength of a Slow Electron** Compute the de Broglie wavelength of an electron whose kinetic energy is 10 eV.

Solution

1. The de Broglie wavelength is given by Equation 5-1:

$$\lambda = \frac{h}{p}$$

2. *Method 1:* Since a 10-eV electron is nonrelativistic, we can use the classical relation connecting the momentum and the kinetic energy:

$$E_k = \frac{p^2}{2m}$$

or

$$p = \sqrt{2mE_k}$$
$$= \sqrt{(2)(9.11 \times 10^{-31}\text{ kg})(10\text{ eV})(1.60 \times 10^{-19}\text{ J/eV})}$$
$$= 1.71 \times 10^{-24}\text{ kg}\cdot\text{m/s}$$

3. Substituting this result into Equation 5-1:

$$\lambda = \frac{6.63 \times 10^{-34}\text{ J}\cdot\text{s}}{1.71 \times 10^{-24}\text{ kg}\cdot\text{m/s}}$$
$$= 3.88 \times 10^{-10}\text{ m} = 0.39\text{ nm}$$

4. *Method 2:* The electron's wavelength can also be computed from Equation 5-4 with $V_0 = 10$ V:

$$\lambda = \frac{1.226}{V^{1/2}} = \frac{1.226}{\sqrt{10}}$$
$$= 0.39\text{ nm}$$

Remarks: *Though this wavelength is small, it is just the order of magnitude of the size of an atom and of the spacing of atoms in a crystal.*

The Davisson-Germer Experiment

In a brief note in the August 14, 1925, issue of the journal *Naturwissenschaften,* Walter Elsasser, at the time a student of Franck's (of the Franck-Hertz experiment), proposed that the wave effects of low-velocity electrons might be detected by scattering them from single crystals. The first such measurements of the wavelengths of electrons were made in 1927 by Davisson[5] and Germer, who were studying electron reflection from a nickel target at Bell Telephone Laboratories, unaware of either Elsasser's suggestion or de Broglie's work. After heating their target to remove an oxide coating that had accumulated during an accidental break in their vacuum system, they found that the scattered electron intensity as a function of the scattering angle showed maxima and minima. Their target had crystallized in the process of cooling, and they were observing electron diffraction. Recognizing the importance of their accidental discovery, they then prepared a target consisting of a single crystal of nickel and extensively investigated the scattering of electrons from it. Figure 5-2 illustrates their experimental arrangement. Their data for 54-eV electrons, shown in Figure 5-3, indicate a strong maximum of scattering at $\phi = 50°$. Consider the scattering from a set of Bragg planes, as shown in Figure 5-4. The Bragg condition for constructive interference is $n\lambda = 2d \sin\theta = 2d\cos\alpha$. The spacing of the Bragg planes d is related to the spacing of the atoms D by $d = D\sin\alpha$; thus

$$n\lambda = 2D \sin\alpha \cos\alpha = D \sin 2\alpha$$

or

$$n\lambda = D\sin\phi \qquad 5\text{-}5$$

where $\phi = 2\alpha$ is the scattering angle.

The spacing D for Ni is known from x-ray diffraction to be 0.215 nm. The wavelength calculated from Equation 5-5 for the peak observed at $\phi = 50°$ by Davisson and Germer is, for $n = 1$,

$$\lambda = 0.215 \sin 50° = 0.165\text{ nm}$$

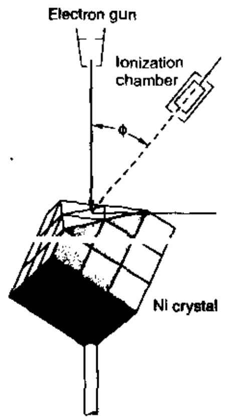

Fig. 5-2 The Davisson-Germer experiment. Low-energy electrons scattered at angle ϕ from a nickel crystal are detected in an ionization chamber. The kinetic energy of the electrons could be varied by changing the accelerating voltage on the electron gun.

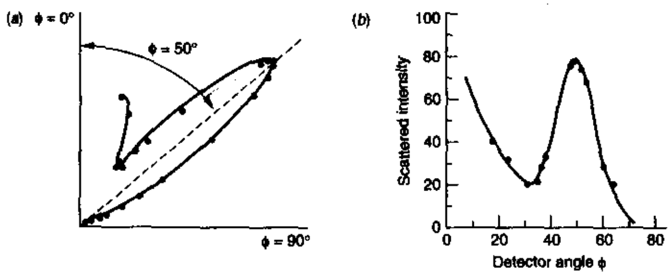

Fig. 5-3 Scattered intensity vs. detector angle for 54-eV electrons. (*a*) Polar plot of the data. The intensity at each angle is indicated by the distance of the point from the origin. Scattering angle ϕ is plotted clockwise starting at the vertical axes. (*b*) The same data plotted on a Cartesian graph. The intensity scales are arbitrary, but the same on both graphs. In each plot there is maximum intensity at $\phi = 50°$, as predicted for Bragg scattering of waves having wavelength $\lambda = h/p$. [*From* Nobel Prize Lectures: Physics *(Amsterdam and New York: Elsevier, © Nobel Foundation, 1964).*]

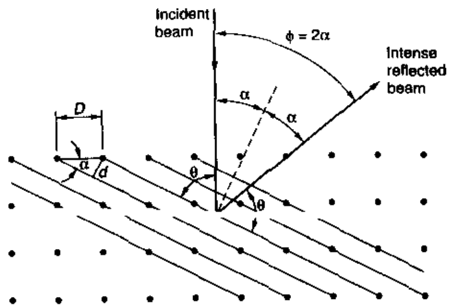

Fig. 5-4 Scattering of electrons by a crystal. Electron waves are strongly scattered if the Bragg condition $n\lambda = 2d \sin\theta$ is met. This is equivalent to the condition $n\lambda = D \sin\phi$.

The value calculated from the de Broglie relation for 54-eV electrons is

$$\lambda = \frac{1.226}{(54)^{1/2}} = 0.167 \text{ nm}$$

The agreement with the experimental observation is excellent! With this spectacular result Davisson and Germer then conducted a systematic study to test the de Broglie relation using electrons up to about 400 eV and various experimental arrangements. Figure 5-5 shows a plot of measured wavelengths versus $V_0^{-1/2}$. The wavelengths measured by diffraction are slightly lower than the theoretical predictions because the refraction of the electron waves at the crystal surface has been neglected. We have seen from the photoelectric effect that it takes work of the order of several eV to remove an electron from a metal. Electrons entering a metal thus gain kinetic energy; therefore, their de Broglie wavelength is slightly less inside the crystal.[6]

A subtle point must be made here. Notice that the wavelength in Equation 5-5 depends only on D, the interatomic spacing of the crystal, whereas our derivation of

Clinton J. Davisson (left) and Lester H. Germer at Bell Laboratories, where electron diffraction was first observed. [*Bell Telephone Laboratories, Inc.*]

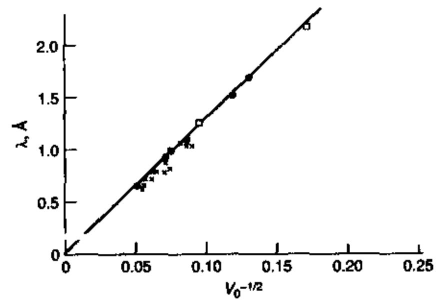

Fig. 5-5 Test of the de Broglie formula $\lambda = h/p$. The wavelength is computed from a plot of the diffraction data plotted against $V_0^{-1/2}$, where V_0 is the accelerating voltage. The straight line is $1.226V_0^{-1/2}$ nm as predicted from $\lambda = h(2mE)^{-1/2}$. These are the data referred to in the quotation from Davisson's Nobel lecture. (× From observations with diffraction apparatus; ⊗ same, particularly reliable; □ same, grazing beams. ⊙ From observations with reflection apparatus.) [*From* Nobel Prize Lectures: Physics *(Amsterdam and New York: Elsevier,* © *Nobel Foundation, 1964).*]

that equation included the interplane spacing as well. The fact that the structure of the crystal really is essential shows up when the energy is varied, as was done in collecting the data for Figure 5-5. Equation 5-5 suggests that a change in λ, resulting from a change in the energy, would mean only that the diffraction maximum would occur at some other value of φ such that the equation remains satisfied. However, as can be seen from examination of Figure 5-4, the value of φ is determined by α, the angle of the planes determined by the crystal structure. Thus, if there are no crystal planes making an angle $\alpha = \phi/2$ with the surface, then setting the detector at $\phi = \sin^{-1}(\lambda/D)$ will not result in constructive interference and strong reflection for that value of λ, even though Equation 5-5 is satisfied. This is neatly illustrated by Figure 5-6, which shows a series of polar graphs (like Figure 5-3*a*) for electrons of energies from 36 eV through 68 eV. The building to a strong reflection at φ = 50° is evident for V_0 = 54 V, as we have already seen. But Equation 5-5 by itself would also lead us to expect, for example, a strong reflection at φ = 64° when V_0 = 40 V, which obviously does not occur.

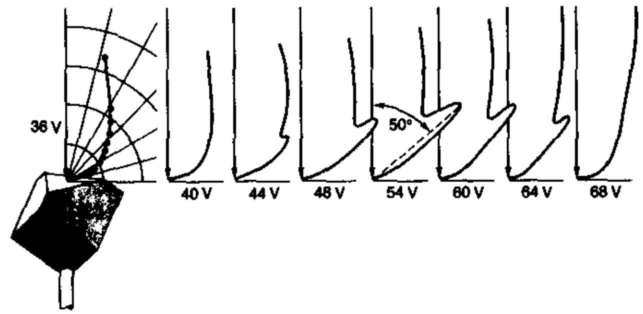

Fig. 5-6 A series of polar graphs of Davisson and Germer's data at electron accelerating potentials from 36 V to 68 V. Note the development of the peak at φ = 50° to a maximum when V_0 = 54 V.

In order to show the dependence of the diffraction on the inner atomic layers, Davisson and Germer kept the detector angle ϕ fixed and varied the accelerating voltage, rather than search for the correct angle for a given λ. Writing Equation 5-5 as

$$\lambda = \frac{D \sin \phi}{n} = \frac{D \sin (2\alpha)}{n} \qquad \text{5-6}$$

and noting that $\lambda \propto V_0^{-1/2}$, a graph of intensity versus $V_0^{-1/2}$ for a given angle ϕ should yield (1) a series of equally spaced peaks corresponding to successive values of the integer n, if $\alpha = \phi/2$ is an existing angle for atomic planes, or (2) no diffraction peaks if $\phi/2$ is not such an angle. Their measurements verified the dependence upon the interplane spacing, the agreement with the prediction being about ± 1 percent. Figure 5-7 illustrates the results for $\phi = 50°$. Thus, Davisson and Germer showed conclusively that particles with mass moving at speeds $v \ll c$ do indeed have wavelike properties, as de Broglie had proposed.

Here is Davisson's account of the connection between de Broglie's predictions and their experimental verification:

The diffraction pattern formed by high-energy electron waves scattered from nuclei provides a means by which nuclear radii and the internal distribution of the nuclear charge (the protons) are measured. See Chapter 11.

> Perhaps no idea in physics has received so rapid or so intensive development as this one. De Broglie himself was in the van of this development, but the chief contributions were made by the older and more experienced Schrödinger. It would be pleasant to tell you that no sooner had Elsasser's suggestion appeared than the experiments were begun in New York which resulted in a demonstration of electron diffraction—pleasanter still to say that the work was begun the day after copies of de Broglie's thesis reached America. The true story contains less of perspicacity and more of chance. . . . It was discovered, purely by accident, that the intensity of elastic scattering [of electrons] varies with the orientations of the scattering crystals. Out of this grew, quite naturally, an investigation of elastic scattering by a single crystal of predetermined orientation. . . . Thus the New York experiment was not, at its inception, a test of wave theory. Only in the summer of 1926, after I had discussed the investigation in England with Richardson, Born, Franck and others, did it take on this character.[7]

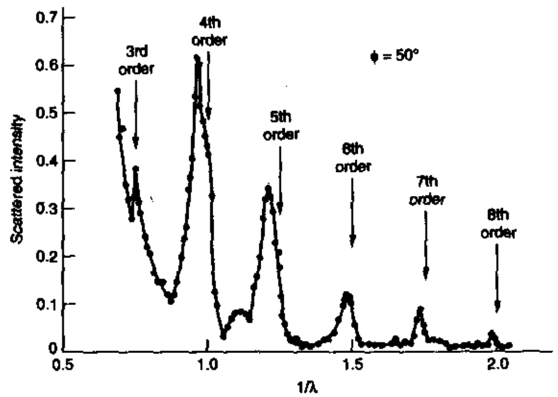

Fig. 5-7 Variation of the scattered electron intensity with wavelength for constant ϕ. The incident beam in this case was 10° from the normal, the resulting refraction causing the measured peaks to be slightly shifted from the positions computed from Equation 5-5, as explained in note 6. [*After C. J. Davisson and L. H. Germer, Proceedings of the National Academy of Sciences, 14, 619 (1928).*]

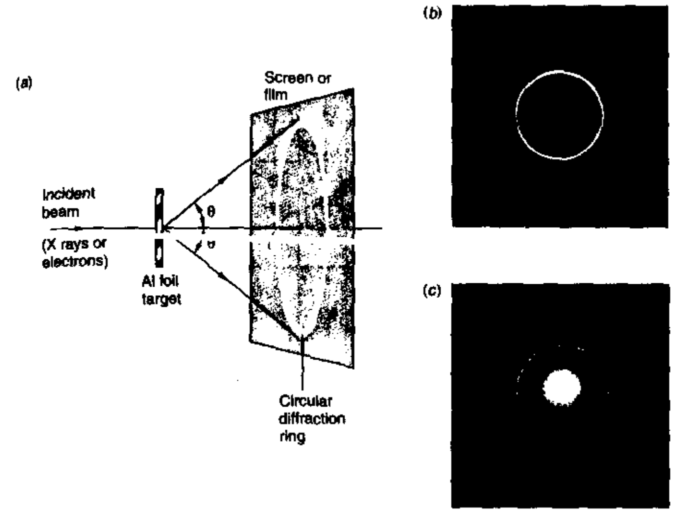

Fig. 5-8 (*a*) Schematic arrangement used for producing a diffraction pattern from a polycrystalline aluminum target. (*b*) Diffraction pattern produced by x rays of wavelength 0.071 nm and an aluminum foil *target*. (*c*) Diffraction pattern produced by 600-eV electrons (de Broglie wavelength of about 0.05 nm) and an aluminum foil target. The pattern has been enlarged by 1.6 times to facilitate comparison with (*b*). [*Courtesy of Film Studio, Education Development Center.*]

A demonstration of the wave nature of relativistic electrons was provided in the same year by G. P. Thomson, who observed the transmission of electrons with energies in the range of 10 to 40 keV through thin metallic foils (G. P. Thomson, the son of J. J. Thomson, shared the Nobel Prize in 1937 with Davisson). The experimental arrangement (Figure 5-8*a*) was similar to that used to obtain Laue patterns with x rays (see Figure 3-14). Because the metal foil consists of many tiny crystals randomly oriented, the diffraction pattern consists of concentric rings. If a crystal is oriented at an angle θ with the incident beam, where θ satisfies the Bragg condition, this crystal will strongly scatter at an equal angle θ; thus there will be a scattered beam making an angle 2θ with the incident beam. Figures 5-8*b* and *c* show the similarities in patterns produced by x rays and electron waves.

Diffraction of Other Particles The wave properties of neutral atoms and molecules were first demonstrated by Stern and Estermann in 1930 with beams of helium atoms and hydrogen molecules diffracted from a lithium fluoride crystal. Since the particles are neutral, there is no possibility of accelerating them with electrostatic potentials. The energy of the molecules was that of their average thermal motion, about 0.03 eV, which implies a de Broglie wavelength of about 0.10 nm for these molecules, according to Equation 5-2. Because of their low energy, the scattering occurs just from the array of atoms on the surface of the crystal, in contrast to Davisson and Germer's experiment. Figure 5-9 illustrates the geometry of the surface scattering, the experimental arrangement, and the results. Figure 5-9*c* indicates clearly the diffraction of He atom waves.

The diffraction patterns formed by helium atom waves are used to study impurities and defects on the surfaces of crystals. Being a noble gas, helium does not react chemically with molecules on the surface or "stick" to the surface.

Since then, diffraction of other atoms, of protons, and of neutrons has been observed (Figures 5-10, 5-11, and 5-12). In all cases the measured wavelengths agree with de Broglie's prediction. There is thus no doubt that all matter has wavelike, as well as particlelike, properties, in symmetry with electromagnetic radiation.

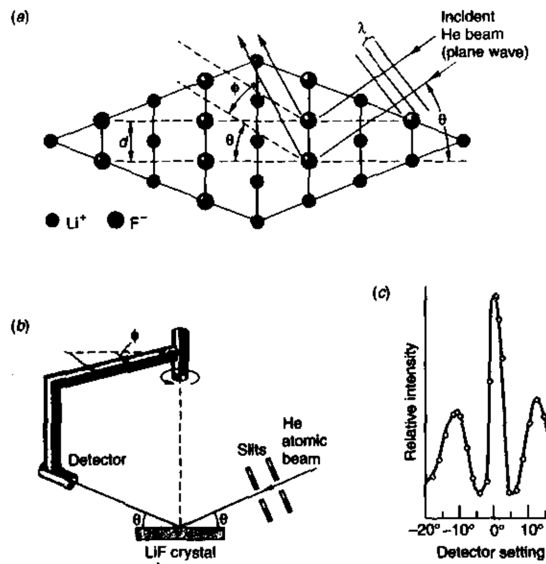

Fig. 5-9 (*a*) He atoms impinge upon the surface of the LiF crystal at angle θ (θ = 18.5° in Estermann and Stern's experiment). The reflected beam also makes the same angle θ with the surface, but is also scattered at azimuthal angles φ relative to an axis perpendicular to the surface. (*b*) The detector views the surface at angle θ but can scan through the angle φ. (*c*) At angle φ where the path difference ($d \sin \phi$) between adjacent "rays" is $n\lambda$, constructive interference, i.e., a diffraction peak, occurs. The $n = 1$ peaks occur on either side of the $n = 0$ maximum.

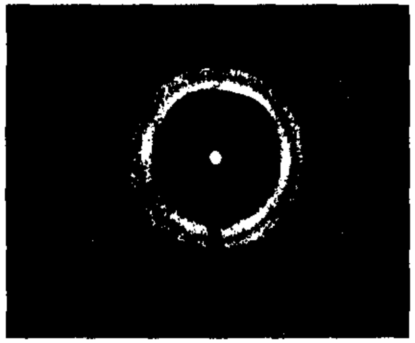

Fig. 5-10 Diffraction pattern produced by 0.0568-eV neutrons (de Broglie wavelength of 0.120 nm) and a target of polycrystalline copper. Note the similarity in the patterns produced by x rays, electrons, and neutrons. [*Courtesy of C. G. Shull.*]

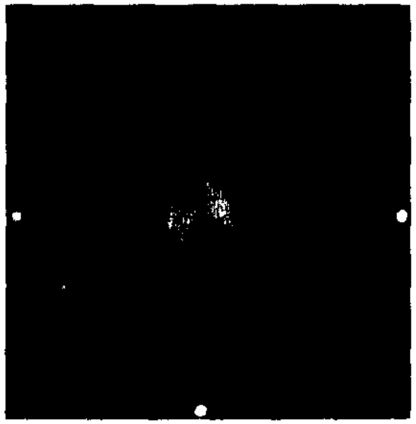

Fig. 5-11 Neutron Laue pattern of NaCl. Compare this with the x-ray Laue pattern in Figure 3-14. [*Courtesy of E. O. Wollan and C. G. Shull.*]

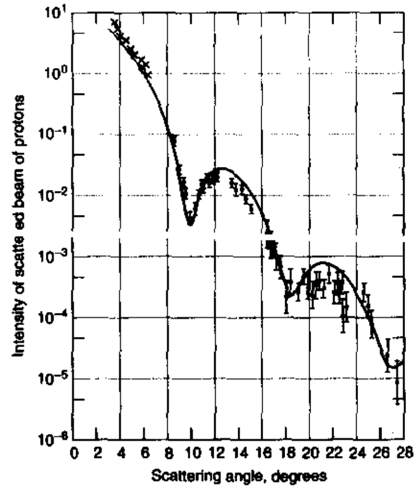

Fig. 5-12 Nuclei provide scatterers whose dimensions are of the order of 10^{-15} m. Here the diffraction of 1-GeV protons from oxygen nuclei results in a pattern similar to that of a single slit.

An Easy Way to Determine de Broglie Wavelengths

It is frequently helpful to know the de Broglie wavelength for particles with a specific kinetic energy. For low energies where relativistic effects can be ignored, the equation leading to Equation 5-4 can be rewritten in terms of the kinetic energy $E_k = \frac{1}{2}mv^2 = p^2/2m$ as follows:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}} \qquad \text{5-7}$$

To find the equivalent expression that covers both relativistic and nonrelativistic speeds, we begin with the relativistic equation relating the total energy to the momentum:

$$E^2 = (pc)^2 + (mc^2)^2 \qquad \text{2-31}$$

Writing E_0 for the rest energy mc^2 of the particle for convenience, this expression becomes

$$E^2 = (pc)^2 + E_0^2 \qquad \text{5-8}$$

Since the total energy $E = E_0 + E_k$, Equation 5-8 becomes

$$(E_0 + E_k)^2 = (pc)^2 + E_0^2$$

which, when solved for p, yields

$$p = \frac{(2E_0E_k + E_k^2)^{1/2}}{c}$$

from which Equation 5-2 gives

$$\lambda = \frac{hc}{(2E_0E_k + E_k^2)^{1/2}} \qquad \textbf{5-9}$$

This can be written in a particularly useful way applicable to any particle of any energy by dividing the numerator and denominator by the rest energy $E_0 = mc^2$ as follows:

$$\lambda = \frac{hc/mc^2}{(2E_0E_k + E_k^2)^{1/2}/E_0} = \frac{h/mc}{[2(E_k/E_0) + (E_k/E_0)^2]^{1/2}}$$

Recognizing h/mc as the Compton wavelength λ_c of the particle of mass m (see Section 3-4 and Equation 3-31), we have that, for any particle,

$$\lambda/\lambda_c = \frac{1}{[2(E_k/E_0) + (E_k/E_0)^2]^{1/2}} \qquad \textbf{5-10}$$

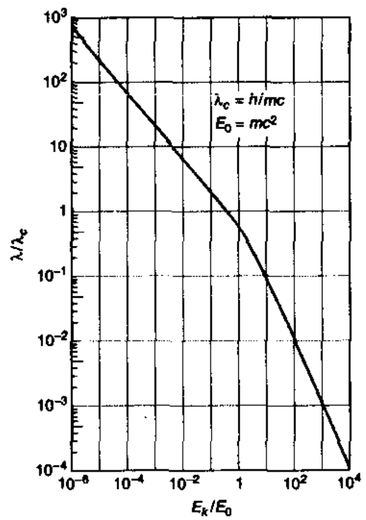

Fig. 5-13 The de Broglie wavelength λ expressed in units of the Compton wavelength λ_c for a particle of mass m versus the kinetic energy of the particle E_k expressed in units of its rest energy $E_0 = mc^2$. For protons and neutrons $E_0 = 0.938$ GeV and $\lambda_c = 1.32$ fm. For electrons $E_0 = 0.511$ MeV and $\lambda_c = 0.00234$ nm.

A log-log graph of λ/λ_c versus E_k/E_0 is shown in Figure 5-13. It has two sections of nearly constant slope, one for $E_k \ll mc^2$ and the other for $E_k \gg mc^2$, connected by a curved portion lying roughly between $0.1 < E_k/E_0 < 10$. The following example illustrates the use of Figure 5-13.

EXAMPLE 5-3 The de Broglie Wavelength of a Cosmic Ray Proton **Detectors on board a satellite measure the kinetic energy of a cosmic ray proton to be 150 GeV. What is the proton's de Broglie wavelength, as read from Figure 5-13?**

Solution

The rest energy of the proton is $mc^2 = 0.938$ GeV and the proton's mass is 1.67×10^{-27} kg. Thus, the ratio E_k/E_0 is

$$\frac{E_k}{E_0} = \frac{150\text{ GeV}}{0.938\text{ GeV}} = 160$$

This value on the curve corresponds to about 2×10^{-3} on the λ/λ_c axis. The Compton wavelength of the proton is

$$\lambda_c = \frac{h}{mc} = \frac{6.63 \times 10^{-34}\text{ J}\cdot\text{s}}{(1.67 \times 10^{-27}\text{ kg})(3 \times 10^8\text{ m/s})} = 1.32 \times 10^{-15}\text{ m}$$

and we have then for the particle's de Broglie wavelength

$$\lambda = (2 \times 10^{-3})(1.32 \times 10^{-15}\text{ m}) = 2.6 \times 10^{-18}\text{ m} = 2.6 \times 10^{-3}\text{ fm}$$

QUESTIONS

1. Since the electrons used by Davisson and Germer were low energy, they penetrated only a few atomic layers into the crystal so it is rather surprising that the effects of the inner layers show so clearly. What feature of the diffraction is most affected by the relatively shallow penetration?
2. How might the frequency of de Broglie waves be measured?
3. Why is it not reasonable to do crystallographic studies with protons?

5-3 Wave Packets

In any discussion of waves the question arises, What's waving? For some waves the answer is clear: for waves on the ocean, it is the water that "waves"; for sound waves in air, it is the molecules that comprise the air; for light, it is the **ℰ** and the **B**. So what is waving for matter waves? As will be developed in this section and the next, for matter it is the *probability of finding the particle* that waves.

Classical waves are solutions of the classical *wave equation*

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad \textbf{5-11}$$

Important among classical waves is the *harmonic wave* of amplitude y_0, frequency f, and period T:

$$y = y_0 \cos(kx - \omega t) = y_0 \cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) = y_0 \cos \frac{2\pi}{\lambda}(x - vt) \qquad \textbf{5-12}$$

where the *angular frequency* ω and the *wave number*[8] k are defined by

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad \textbf{5-13a}$$

and

$$k = \frac{2\pi}{\lambda} \qquad \textbf{5-13b}$$

and the *wave or phase velocity* v_p is given by

$$v_p = f\lambda \qquad \textbf{5-14}$$

Fig. 5-14 Wave pulse moving along a string. A pulse has a beginning and an end; i.e., it is localized, unlike a pure harmonic wave, which goes on forever in space and time.

A familiar wave phenomenon which cannot be described by a single harmonic wave is a pulse, such as the flip of one end of a long string (Figure 5-14), a sudden noise, or the brief opening of a shutter in front of a light source. The main characteristic of a pulse is localization in time and space. A single harmonic wave is not localized in either time or space. The description of a pulse can be obtained by the superposition of a group of harmonic waves of different frequencies and wavelengths. Such a group is called a *wave packet*. The mathematics of representing arbitrarily shaped pulses by sums of sine or cosine functions involves Fourier series and Fourier integrals. We shall illustrate the phenomenon of wave packets by considering some simple and somewhat artificial examples and discussing the general properties qualitatively. Wave groups are particularly important because a wave description of a particle must include the important property of localization.

Consider a simple group consisting of only two waves of equal amplitude and nearly equal frequencies and wavelengths. Such a group occurs in the phenomenon of beats and is described in most introductory textbooks. Let the wave numbers be k_1 and k_2, the angular frequencies ω_1 and ω_2, and the speeds v_1 and v_2. The sum of the two waves is

$$y(x, t) = y_0 \cos(k_1 x - \omega_1 t) + y_0 \cos(k_2 x - \omega_2 t)$$

which, with the use of a bit of trigonometry, becomes

$$y(x, t) = 2y_0 \cos\left(\frac{\Delta k}{2}x - \frac{\Delta w}{2}t\right)\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)$$

where $\Delta k = k_2 - k_1$ and $\Delta\omega = \omega_2 - \omega_1$. Since the two waves have nearly equal values of k and ω, we will write $\bar{k} = (k_1 + k_2)/2$ and $\bar{\omega} = (\omega_1 + \omega_2)/2$ for the mean values. The sum is then

$$y(x, t) = 2y_0 \cos\left(\tfrac{1}{2}\Delta k x - \tfrac{1}{2}\Delta\omega t\right)\cos(\bar{k}x - \bar{\omega}t) \qquad \textbf{5-15}$$

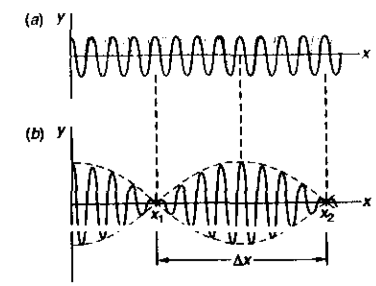

Fig. 5-15 Two waves of slightly different wavelength and frequency produce beats. (*a*) Shows $y(x)$ at a given instant for each of the two waves. The waves are in phase at the origin but because of the difference in wavelength, they become out of phase and then in phase again. (*b*) The sum of these waves. The spatial extent of the group Δx is inversely proportional to the difference in wave numbers Δk, where k is related to the wavelength by $k = 2\pi/\lambda$. Identical figures are obtained if y is plotted versus time t at a fixed point x. In that case the extent in time Δt is inversely proportional to the frequency difference $\Delta\omega$.

Figure 5-15 shows a sketch of $y(x, t_0)$ versus x at some time t_0. The dashed curve is the envelope of the group of two waves, given by the first cosine term in Equation 5-15. The wave within the envelope moves with the speed $\bar{\omega}/\bar{k}$, the phase velocity v_p due to the second cosine term. If we write the first (amplitude modulating) term as $\cos\{\frac{1}{2}\Delta k[x - (\Delta\omega/\Delta k)t]\}$, we see that the envelope moves with speed $\Delta\omega/\Delta k$. The speed of the envelope is called the *group velocity* v_g.

Classical Uncertainty Relations

The range of wavelengths or frequencies of the harmonic waves needed to form a wave packet depends on the extent in space and duration in time of the pulse. In general, if the extent in space Δx is small, the range Δk of wave numbers must be large. Similarly, if the duration in time t is small, the range of frequencies $\Delta\omega$ must be large. It can be shown that for a general wave packet, Δx and Δk are related by

$$\Delta k\, \Delta x \sim 1 \qquad \textbf{5-16}$$

Similarly,

$$\Delta\omega\, \Delta t \sim 1 \qquad \textbf{5-17}$$

The classical uncertainty relations define the range of signal frequencies to which all kinds of communications equipment and computer systems must respond, from cell phones to supercomputers.

We have written these as order-of-magnitude equations because the exact value of the products $\Delta x\, \Delta k$ and $\Delta t\, \Delta\omega$ depends on how these ranges are defined, as well as on the particular shape of the packets. Equation 5-17 is sometimes known as the *response time–bandwidth* relation, expressing the result that a circuit component such as an amplifier must have a large bandwidth ($\Delta\omega$) if it is to be able to respond to signals of short duration.

There is a slight variation of Equation 5-16 that is also helpful in interpreting the relation between Δx and Δk. Differentiating the wave number in Equation 5-13*b* yields

$$dk = \frac{-2\pi d\lambda}{\lambda^2} \qquad \textbf{5-18}$$

Replacing the differentials by small intervals and concerning ourselves only with magnitudes, Equation 5-18 becomes

$$\Delta k = \frac{2\pi\Delta\lambda}{\lambda^2}$$

which when substituted into Equation 5-16 gives

$$\Delta x\, \Delta\lambda \approx \frac{\lambda^2}{2\pi} \qquad \textbf{5-19}$$

Equation 5-19 says that the product of the spatial extent of a classical wave Δx and the uncertainty (or "error") in the determination of its wavelength $\Delta\lambda$ will always be of the order of $\lambda^2/2\pi$. The following brief examples will illustrate the meaning of Equations 5-16 and 5-17, often referred to as the *classical uncertainty relations,* and Equation 5-19.

EXAMPLE 5-4 $\Delta\lambda$ for Ocean Waves **Standing in the middle of a 20-m-long pier, you notice that at any given instant there are 15 wave crests between the two ends of the pier. Estimate the minimum uncertainty in the wavelength that could be computed from this information.**

Solution

1. The minimum uncertainty $\Delta\lambda$ in the wavelength is given by Equation 5-19:

$$\Delta x\, \Delta\lambda \approx \frac{\lambda^2}{2\pi}$$

2. The wavelength λ of the waves is:

$$\lambda = \frac{20 \text{ m}}{15 \text{ waves}} = 1.3 \text{ m}$$

3. The spatial extent of the waves used for this calculation is:

$$\Delta x = 20 \text{ m}$$

4. Solving Equation 5-19 for $\Delta\lambda$ and substituting these values gives:

$$\Delta\lambda \approx \frac{\lambda^2}{2\pi\Delta x} = \frac{(1.3)^2}{2\pi \times 20} = 0.013\text{m}$$

$$\Delta\lambda \approx 0.01\text{m} = 1 \text{ cm}$$

Remarks: This is the minimum *uncertainty. Any error that may exist in the measurement of the number of wave crests would add further uncertainty to the determination of* λ.

EXAMPLE 5-5 Frequency Control **The frequency of the alternating voltage produced at electric generating stations is carefully maintained at 60.00 Hz. The frequency is monitored on a digital frequency meter in the control room. For how long must the frequency be measured and how often can the display be updated, if the reading is to be accurate to within 0.01 Hz?**

Solution

Since $\omega = 2\pi f$, then $\Delta\omega = 2\pi\Delta f = 2\pi(0.01)$ rad/s and

$$\Delta t \sim 1/\Delta\omega = 1/2\pi(0.01)$$

$$\Delta t \sim 16 \text{ s}$$

Thus, the frequency must be measured for about 16 s if the reading is to be accurate to 0.01 Hz and the display cannot be updated more often than once every 16 s.

General Wave Packet

We can construct a more general wave packet than Equation 5-15 if we allow the amplitudes of the various harmonic waves to be different. Such a packet can be represented by an equation of the form

$$y(x, t) = \sum_i A_i \cos(k_i x - \omega_i t) \qquad \textbf{5-20}$$

where A_i is the amplitude of the wave with wave number k_i and angular frequency ω_i. The calculation of the amplitudes A_i needed to construct a wave packet of some given shape $y(x, t_0)$ at some particular time is a problem in Fourier series.

If we are restricted to a finite number of waves, it is not possible to obtain a wave packet that is small everywhere outside a well-defined range. The larger the number of waves, the larger the region in which destructive interference makes the envelope small, but eventually all the waves will again be in phase, the envelope will be large, and the pattern will repeat. To represent a pulse that is zero everywhere outside some range, such as that shown in Figure 5-14, we must construct a wave packet from a continuous distribution of waves. We can do this by replacing A_i in Equation 5-20 by $A(k)\, dk$ and changing the sum to an integral. The quantity $A(k)$ is called the distribution function for the wave number k. Either the shape of the wave packet at some fixed time $y(x)$ or the distribution of wave numbers $A(k)$ can be found from the other by methods of Fourier analysis.[9]

Figure 5-16 shows a Gaussian-shaped wave packet and the corresponding wave-number distribution function for a narrow packet (Figure 5-16*a*) and a wide packet (Figure 5-16*b*). For this special case, $A(k)$ is also a Gaussian function. The standard deviations of these Gaussian functions are related by

$$\sigma_x \sigma_k = \frac{1}{2} \qquad \textbf{5-21}$$

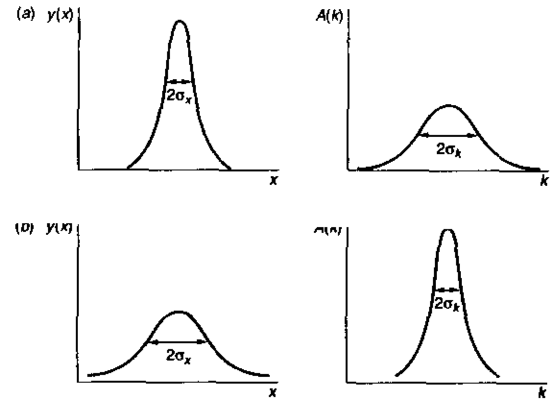

Fig. 5-16 Gaussian-shaped wave packets $y(x)$ and the corresponding Gaussian distributions of wave numbers $A(k)$. (*a*) A narrow packet. (*b*) A wide packet. The standard deviations in each case are related by $\sigma_x \sigma_k = 1/2$.

It can be shown that the product of the standard deviations is greater than 1/2 for a wave packet of any shape other than Gaussian.

For our simple group of only two waves, we found that the envelope moved with the velocity $v_g = \Delta\omega/\Delta k$. For a general wave packet, the group velocity is given by

$$v_g = \frac{d\omega}{dk} \qquad \textbf{5-22}$$

where the derivative is evaluated at the central wave number. The group velocity of a pulse can be related to the phase velocities of the individual harmonic waves making up the packet. The phase velocity of a harmonic wave is

$$v_p = f\lambda = \left(\frac{\omega}{2\pi}\right)\left(\frac{2\pi}{k}\right) = \frac{\omega}{k}$$

so that

$$\omega = kv_p$$

Differentiating and substituting $d\omega/dk$ from Equation 5-22, we obtain

$$v_g = v_p + k\frac{dv_p}{dk} \qquad \textbf{5-23}$$

If the phase velocity is the same for all frequencies and wavelengths, then $dv_p/dk = 0$, and the group velocity is the same as the phase velocity. A medium for which the phase velocity is the same for all frequencies is said to be *nondispersive*. Examples are waves on a perfectly flexible string, sound waves in air, and electromagnetic waves in a vacuum. An important characteristic of a nondispersive medium is that, since all the harmonic waves making up a packet move with the same speed, the packet maintains its shape as it moves; thus it does not change its shape in time.

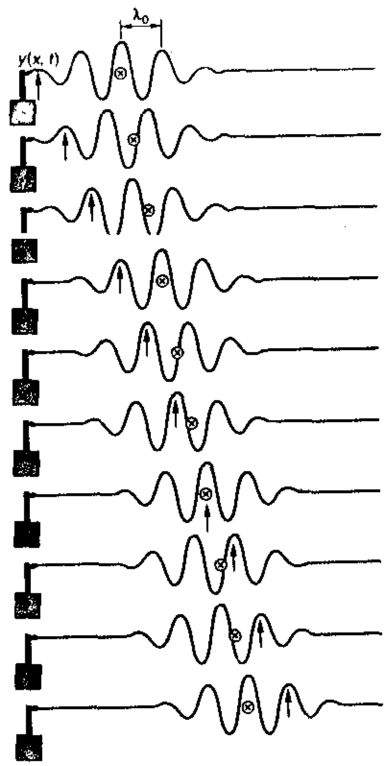

Fig. 5-17 Wave packet for which the group velocity is half the phase velocity. Water waves whose wavelengths are a few centimeters, but much less than the water depth, have this property. The arrow travels at the phase velocity, following a point of constant phase for the dominant wavelength. The cross at the center of the group travels at the group velocity. [*Adapted from F. S. Crawford, Jr.*, Berkeley Physics Course *(New York: McGraw-Hill, 1965), vol. 3, p. 294. Courtesy of Education Development Center, Inc., Newton, Mass.*]

Conversely, if the phase velocity is different for different frequencies, the shape of the pulse will change as it travels. In that case, the group velocity and phase velocity are not the same. Such a medium is called a *dispersive* medium; examples are water waves, waves on a wire that is not perfectly flexible, light waves in a medium such as glass or water, in which the index of refraction has a slight dependence on frequency, and electron waves. Figure 5-17 shows a wave packet for which the group velocity is half the phase velocity. The following example also illustrates such a case.

EXAMPLE 5-6 Velocity of Deep Ocean Waves The phase velocity v of waves deep in the ocean is given by $v_p = gT/2\pi$, where g is the acceleration of gravity and T is the period of the wave. What would be the group velocity of a wave packet formed by a group of such waves, expressed in terms of the phase velocity?

Solution

Note that $v_p = gT/2\pi = g/2\pi f = g/\omega$ from the definitions of the period and the angular velocity. Thus,

$$v_p = \frac{\omega}{k} = \frac{g}{\omega}$$

or

$$gk = \omega^2$$

Since the group velocity $v_g = d\omega/dk$ (Equation 5-22), we differentiate the above expression, obtaining

$$g\,dk = 2\omega\,d\omega$$

or

$$v_g = \frac{d\omega}{dk} = \frac{g}{2\omega} = \frac{1}{2}v_p$$

QUESTIONS

4. Which is more important for communication, the group velocity or the phase velocity?
5. What are Δx and Δk for a purely harmonic wave of a single frequency and wavelength?

Particle Wave Packets

The quantity analogous to the displacement $y(x, t)$ for waves on a string, to the pressure $P(x, t)$ for a sound wave, or to the electric field $\mathcal{E}(x, t)$ for electromagnetic waves, is called the *wave function* for particles and is usually designated $\Psi(x, t)$. It is $\Psi(x, t)$ that we will relate to the probability of finding the particle and, as we alerted you earlier, it is the probability that waves. Consider, for example, an electron wave consisting of a single frequency and wavelength; we could represent such a wave by any of the following, exactly as we did the classical wave: $\Psi(x, t) = A\cos(kx - \omega t)$, $\Psi(x, t) = A\sin(kx - \omega t)$, or $\Psi(x, t) = Ae^{i(kx - \omega t)}$.

The phase velocity is given by

$$v_p = f\lambda = \left(\frac{E}{h}\right)\left(\frac{h}{p}\right) = \frac{E}{p}$$

where we have used the de Broglie relations for the wavelength and frequency. Using the nonrelativistic expression for the energy of a particle moving at speed v in free space (i.e., no potential energy) with no forces acting upon it,

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

we see that the phase velocity is

$$v_p = \frac{E}{p} = \frac{p^2/2m}{p} = \frac{p}{2m} = \frac{v}{2}$$

i.e., the phase velocity of the wave is half the velocity v of an electron with momentum p. The phase velocity does *not* equal the particle velocity. Moreover, a wave of a single frequency and wavelength is not localized but is spread throughout space, which makes it difficult to see how the particle and wave properties of the electron could be related. Thus, for the electron to have the particle property of being localized, the matter waves of the electron must also be limited in spatial extent—i.e., realistically, $\Psi(x, t)$ must be a wave packet containing many more than one wave number k and frequency ω. It is the wave packet $\Psi(x, t)$ that we expect to move at a velocity equal to the particle velocity, which we will show below is indeed the case. The particle, if observed, we will expect to find somewhere within the spatial extent of the wave packet $\Psi(x, t)$, precisely where within that space being the subject of the next section.

To illustrate the equality of the group velocity v_g and the particle velocity v it is convenient to express de Broglie's relations in a slightly different form. Writing Equation 5-1 as follows,

An application of phase and particle speeds by nature: produce a wave on a still pond (or in a bathtub) and watch the wavelets that make up the wave appear to "climb over" the wave crest at twice the speed of the wave.

$$E = hf = \frac{h\omega}{2\pi} \quad \text{or} \quad E = \hbar\omega \qquad \textbf{5-24}$$

and Equation 5-2 as

$$p = \frac{h}{\lambda} = \frac{h}{2\pi/k} = \frac{hk}{2\pi} \quad \text{or} \quad p = \hbar k \qquad \textbf{5-25}$$

The group velocity is then given by

$$v_g = \frac{d\omega}{dk} = \frac{dE/\hbar}{dp/\hbar} = \frac{dE}{dp}$$

Again using the nonrelativistic expression $E = p^2/2m$, we have that

$$v_g = \frac{dE}{dp} = \frac{p}{m} = v$$

and the wave packet $\Psi(x, t)$ moves with the velocity of the electron. This was, in fact, one of de Broglie's reasons for choosing Equations 5-1 and 5-2. (De Broglie used the relativistic expression relating energy and momentum, which also leads to the equality of the group velocity and particle velocity.)

5-4 The Probabilistic Interpretation of the Wave Function

Let us consider in more detail the relation between the wave function $\Psi(x, t)$ and the location of the electron. We can get a hint about this relation from the case of light. The wave equation that governs light is Equation 5-11, with $y = \mathcal{E}$, the electric field, as the wave function. The energy per unit volume in a light wave is proportional to $\mathscr{E}^2$, but the energy in a light wave is quantized in units of hf for each photon. We expect, therefore, that the number of photons in a unit volume is proportional to $\mathscr{E}^2$, a connection first pointed out by Einstein.

Consider the famous double-slit interference experiment (Figure 5-18). The pattern observed on the screen is determined by the interference of the waves from the slits. At a point on the screen where the wave from one slit is 180° out of phase with that from the other, the resultant electric field is zero; there is no light energy at this point, and the point is dark. If we reduce the intensity to a very low value, we can still observe the interference pattern if we replace the ordinary screen by a scintillation screen or a two-dimensional array of tiny photon detectors and wait a sufficient length of time.

The interaction of light with the detector or scintillator is a quantum phenomenon. If we illuminate the scintillators or detectors for only a very short time with a low-intensity source, we do not see merely a weaker version of the high-intensity pattern; we see, instead, "dots" caused by the interactions of individual photons (see Figure 5-19). At points where the waves from the slits interfere destructively there are no dots, and at points where the waves interfere constructively there are many dots. However, when the exposure is short and the source weak, random fluctuations from the average predictions of the wave theory are clearly evident. If the exposure is long enough that many photons reach the detector, the fluctuations average out and the quantum nature of light is not noticed. The interference pattern depends only on the total number of photons interacting with the detector and not on the rate. Even when the intensity is so low that only one photon at a time reaches the detector, the wave theory predicts the correct average pattern. For low intensities, we therefore interpret $\mathscr{E}^2$ as proportional to the *probability* of detecting a photon in a unit volume of space. At points on the detector where $\mathscr{E}^2$ is zero, photons are never observed, whereas they are most likely to be observed at points where $\mathscr{E}^2$ is large.

It is not necessary to use light waves to produce an interference pattern. Such patterns can be produced with electrons and other particles as well. In the wave theory of electrons, the de Broglie wave of a *single* electron is described by a wave function Ψ. The amplitude of Ψ at any point is related to the probability of finding the particle at that point. In analogy with the foregoing interpretation of $\mathscr{E}^2$, the quantity $|\Psi|^2$ is proportional to the probability of detecting an electron in a unit volume, where $|\Psi|^2 = \Psi^*\Psi$,

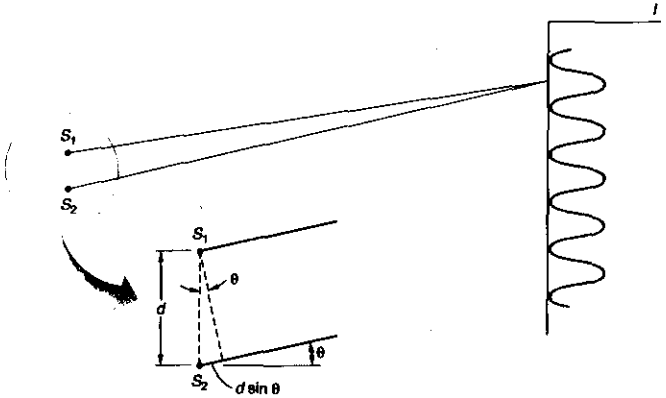

Fig. 5-18 Two-source interference pattern. If the sources are coherent and in phase, the waves from the sources interfere constructively at points for which the path difference ($d \sin \theta$) is an integral number of wavelengths.

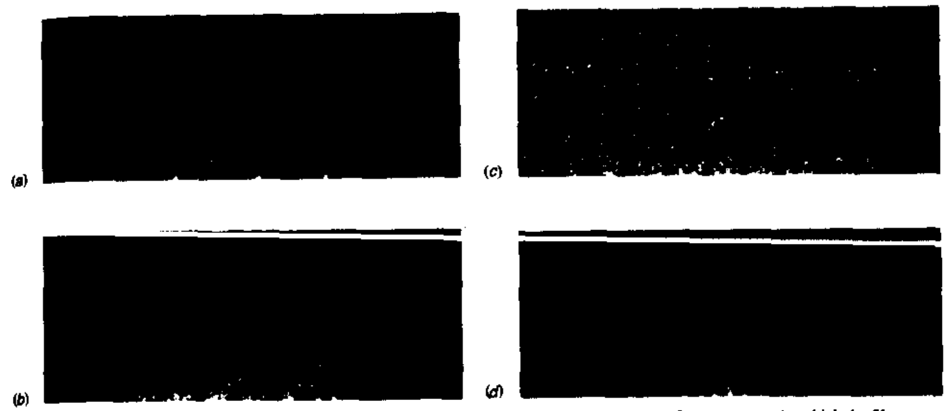

Fig. 5-19 Growth of two-slit interference pattern. The photo (*d*) is an actual two-slit electron interference pattern in which the film was exposed to millions of electrons. The pattern is identical to that usually obtained with photons. If the film were to be observed at various stages, such as after being struck by 28 electrons, then after about 1000 electrons, and again after about 10,000 electrons, the patterns of individually exposed grains would be similar to those shown in (*a*), (*b*), and (*c*), except that the exposed dots would be smaller than the dots drawn here. Note that there are no dots in the region of the interference minima. The probability of any point of the film being exposed is determined by wave theory, whether the film is exposed by electrons or photons. [*Parts (a), (b), and (c) from E. R. Huggins,* Physics 1, *© by W. A. Benjamin, Inc., Menlo Park, California. Photo (d) courtesy of C. Jonsson.*]

the function Ψ^* being the complex conjugate of Ψ. In one dimension, $|\Psi|^2 dx$ is the probability of an electron being in the interval dx.[10] (See Figure 5-20.) If we call this probability $P(x)dx$, where $P(x)$ is the probability distribution function, we have

$$P(x)\,dx = |\Psi|^2\,dx \qquad \text{5-26}$$

In the next chapter we will discuss more thoroughly the amplitudes of matter waves associated with particles, in particular developing the mathematical system for computing the amplitudes and probabilities in various situations. The uneasiness that you may feel at this point regarding the fact that we have not given a precise physical interpretation to the amplitude of the de Broglie matter wave can be attributed in part to the complex nature of the wave amplitude, i.e., it is in general a complex quantity with a real part and an imaginary part, the latter proportional to $i = (-1)^{1/2}$. We cannot directly measure or physically interpret complex numbers in our world of real numbers. However, as we will see, defining the probability in terms of $|\Psi|^2$, which is always real, presents no difficulty in its physical interpretation. Thus, even though the amplitudes of the wave functions Ψ have no simple meaning, the waves themselves behave just as do classical waves, exhibiting the wave characteristics of reflection, refraction, interference, and diffraction and obeying the principles of superposition.

5-5 The Uncertainty Principle

Consider a wave packet $\Psi(x, t)$ representing an electron. The most probable position of the electron is the value of x for which $|\Psi(x, t)|^2$ is a maximum. Since $|\Psi(x, t)|^2$ is proportional to the probability that the electron is at x, and $|\Psi(x, t)|^2$

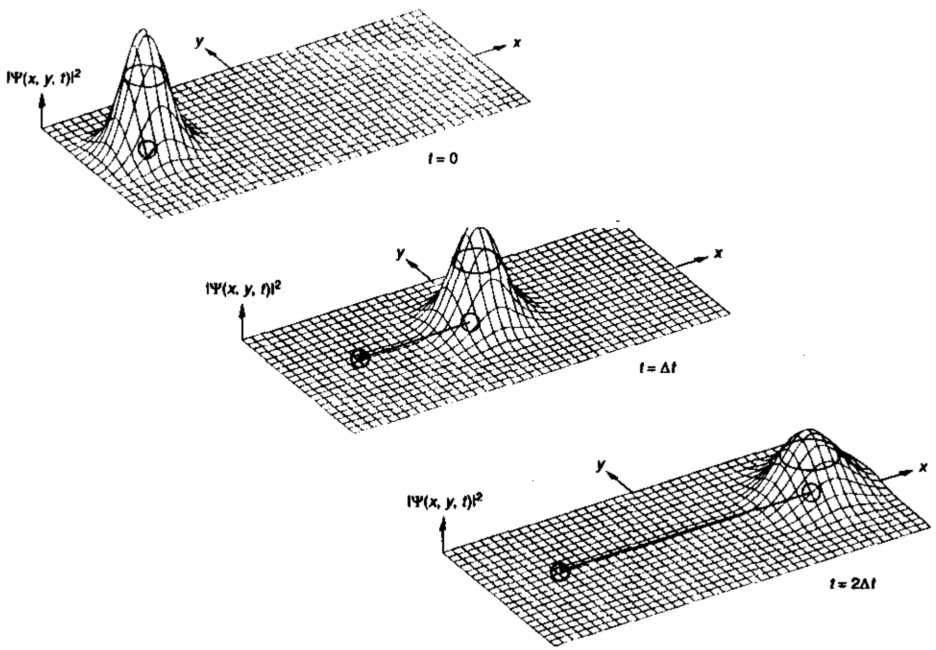

Fig. 5-20 A three-dimensional wave packet representing a particle moving along the x axis. The dot indicates the position of a classical particle. Note that the packet spreads out in the x and y directions. This spreading is due to dispersion, resulting from the fact that the phase velocity of the individual waves making up the packet depends on the wavelength of the waves.

is nonzero for a range of values of x, there is an *uncertainty* in the value of the position of the electron. (See Figure 5-20.) This means that if we make a number of position measurements on identical electrons—electrons with the same wave function—we shall not always obtain the same result. In fact, the distribution function for the results of such measurements will be given by $|\Psi(x, t)|^2$. If the wave packet is very narrow, the uncertainty in position will be small. However, a narrow wave packet must contain a wide range of wave numbers k. Since the momentum is related to the wave number by $p = \hbar k$, a wide range of k values means a wide range of momentum values. We have seen that for all wave packets the ranges Δx and Δk are related by

$$\Delta x \, \Delta k \sim 1 \qquad \textbf{5-16}$$

Similarly, a packet that is localized in time Δt must contain a range of frequencies $\Delta\omega$, where the ranges are related by

$$\Delta\omega \, \Delta t \sim 1 \qquad \textbf{5-17}$$

Equations 5-16 and 5-17 are inherent properties of waves. If we multiply these equations by $\hbar$ and use $p = \hbar k$ and $E = \hbar\omega$, we obtain

$$\Delta x \, \Delta p \sim \hbar \qquad \textbf{5-27}$$

and

$$\Delta E \, \Delta t \sim \hbar \qquad \textbf{5-28}$$

Equations 5-27 and 5-28 provide a statement of the *uncertainty principle* first enunciated in 1927 by Werner K. Heisenberg.[11] Equation 5-27 expresses the fact that the distribution functions for position and momentum cannot both be made arbitrarily narrow simultaneously (see Figure 5-16); thus measurements of position and momentum will have similar uncertainties which are related by Equation 5-27. Of course, because of inaccurate measurements, the product of Δx and Δp can be, and usually is, much larger than $\hbar$. The lower limit is not due to any technical problem in the design of measuring equipment that might be solved at some later time; it is instead due to the wave and particle nature of both matter and light.

If we define precisely what we mean by the uncertainty in the measurements of position and momentum, we can give a precise statement of the uncertainty principle. We saw in Section 5-3 that, if σ_x is the standard deviation for measurements of position and σ_k is the standard deviation for measurements of the wave number k, the product $\sigma_x\sigma_k$ has its minimum value of 1/2 when the distribution functions are Gaussian. If we define Δx and Δp to be the standard deviations, the minimum value of their product is $\frac{1}{2}\hbar$. Thus

$$\Delta x \, \Delta p \geq \tfrac{1}{2}\hbar \qquad \textbf{5-29}$$

Heisenberg's uncertainty principle is the key to the existence of *virtual particles* that hold the nuclei together (see Chapter 11) and is the root of quantum fluctuations that may have been the origin of the Big Bang (see Chapter 14).

Similarly,

$$\Delta E \, \Delta t \geq \tfrac{1}{2}\hbar \qquad \textbf{5-30}$$

QUESTION

6. Does the uncertainty principle say that the momentum of a particle can never be precisely known?

Exploring

The Gamma-Ray Microscope

Let us see how one might attempt to make a measurement so accurate as to violate the uncertainty principle. A common way to measure the position of an object such as an electron is to look at it with light, i.e., scatter light from it and observe the diffraction pattern. The momentum can be obtained by looking at it again, a short time later, and computing what velocity it must have had the instant before the light scattered from it. Because of diffraction effects, we cannot hope to make measurements of length (position) that are smaller than the wavelength of the light used, so we will use the shortest-wavelength light that can be obtained, gamma rays. (There is, in principle, no limit to how short the wavelength of electromagnetic radiation can be.) We also know that light carries momentum and energy, so that when it scatters off the electron, the motion of the electron will be disturbed, affecting the momentum. We must, therefore, use the minimum intensity possible, so as to disturb the electron as little as possible. Reducing the intensity decreases the number of photons, but we must scatter at least one photon to observe the electron. The minimum possible intensity, then, is that corresponding to one photon. The scattering of a photon by a free electron is, of course, a Compton scattering, which was discussed in Section 3-4. The momentum of the photon is $hf/c = h/\lambda$. The smaller the λ that is used to measure the position, the more the photon will disturb the electron, but we can correct for that with a Compton-effect analysis, provided only that we know the photon's momentum and the scattering angles of the event.

Figure 5-21 illustrates the problem. (This illustration was first given as a gedankenexperiment, or thought experiment, by Heisenberg. Since a single photon doesn't form a diffraction pattern, think of the diffraction pattern as being built up by photons from many identical scattering experiments.) The position of the electron is to be determined by viewing it through a microscope. We shall assume that only one photon is used. We can take for the uncertainty in position the minimum separation distance for which two objects can be resolved; this is[12]

$$\Delta x = \frac{\lambda}{2 \sin \theta}$$

where θ is the half angle subtended by the lens aperture, as shown in Figures 5-21*a* and *b*. Let us assume that the x component of momentum of the incoming photon is known precisely from a previous measurement. To reach the screen and contribute to the diffraction pattern in Figure 5-21*c*, the scattered photon need only go through the lens aperture. Thus, the scattered photon can have any x component of momentum from 0 to $p_x = p \sin \theta$, where p is the total momentum of the scattered photon. By conservation of momentum, the uncertainty in the momentum of the electron after the scattering must be greater than or equal to that of the scattered photon (it would be equal, of course, if the electron's initial momentum were known precisely); thus we write

$$\Delta p_x \geq p \sin \theta = \frac{h}{\lambda} \sin \theta$$

and

$$\Delta x \, \Delta p_x \geq \frac{\lambda}{2 \sin \theta} \frac{h \sin \theta}{\lambda} = \frac{1}{2} h$$

Thus, even though the electron prior to our observation may have had a definite position and momentum, our observation has unavoidably introduced an uncertainty in the measured values of those quantities. This illustrates the essential point of the uncertainty principle—that this product of uncertainties cannot be less than about $\hbar$ *in principle*, that is, even in an ideal situation. If electrons rather than photons were used to locate the object, the analysis would not change, since the relation $\lambda = h/p$ is the same for both.

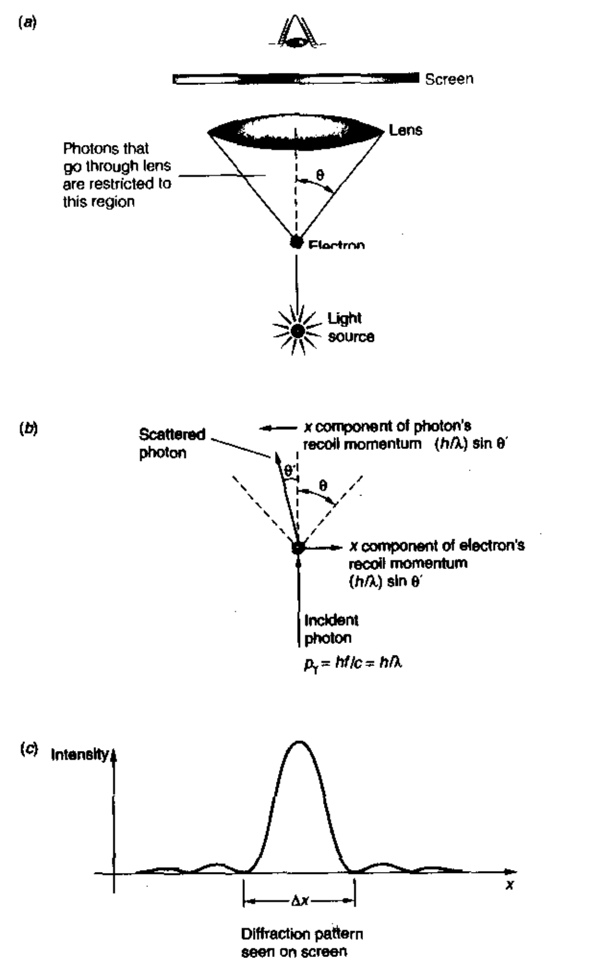

Fig. 5-21 (*a*) "Seeing an electron" with a gamma-ray microscope. (*b*) Because of the size of the lens, the momentum of the scattered photon is uncertain by $\Delta p_x \approx p \sin \theta = h \sin \theta/\lambda$. Thus the recoil momentum of the electron is also uncertain by at least this amount. (*c*) The position of the electron cannot be resolved better than the width of the central maximum of the diffraction pattern $\Delta x \approx \lambda/\sin \theta$. The product of the uncertainties $\Delta p_x \Delta x$ is therefore of the order of Planck's constant h.

5-6 Some Consequences of the Uncertainty Principle

In the next chapter we shall see that the Schrödinger wave equation provides a straightforward method of solving problems in atomic physics. However, the solution of the Schrödinger equation is often laborious and difficult. Much semiquantitative information about the behavior of atomic systems can be obtained from the uncertainty principle alone without a detailed solution of the problem. The general approach used in applying the uncertainty principle to such systems will first be illustrated by considering a particle moving in a box with rigid walls. We then use that analysis in several numerical examples and as a basis for discussing some additional consequences.

Minimum Energy of a Particle in a Box

An important consequence of the uncertainty principle is that a particle confined to a finite space cannot have zero kinetic energy. Let us consider the case of a one-dimensional "box" of length L. If we know that the particle is in the box, Δx is not larger than L. This implies that Δp is at least $\hbar/L$. (Since we are interested in orders of magnitude, we shall ignore the 1/2 in the minimum uncertainty product. In general, distributions are not Gaussian anyway, so $\Delta p\, \Delta x$ will be larger than $\frac{1}{2}\hbar$.) Let us take the standard deviation as a measure of Δp,

$$(\Delta p)^2 = (p - \bar{p})^2_{av} = (p^2 - 2p\bar{p} + \bar{p}^2)_{av} = \overline{p^2} - \bar{p}^2$$

If the box is symmetric, $\bar{p}$ will be zero since the particle moves to the left as often as to the right. Then

$$(\Delta p)^2 = \overline{p^2} \geq \left(\frac{\hbar}{L}\right)^2$$

and the average kinetic energy is

$$\bar{E} = \frac{\overline{p^2}}{2m} \geq \frac{\hbar^2}{2mL^2} \qquad \textbf{5-31}$$

Thus, we see that the uncertainty principle indicates that the minimum energy of a particle (*any* particle) in a "box" (*any* kind of "box") cannot be zero. This minimum energy given by Equation 5-31 for a particle in a one-dimensional box is called the *zero-point energy.*

EXAMPLE 5-7 A Macroscopic Particle in a Box **Consider a small but macroscopic particle of mass $m = 10^{-6}$ g confined to a one-dimensional box with $L = 10^{-6}$ m, e.g., a tiny bead on a very short wire. Compute the bead's minimum kinetic energy and the corresponding speed.**

Solution

1. The minimum kinetic energy is given by Equation 5-31:

$$\bar{E} = \frac{\hbar^2}{2mL^2} = \frac{(1.055 \times 10^{-34}\,\text{J}\cdot\text{s})^2}{(2)(10^{-9}\,\text{kg})(10^{-6}\,\text{m})^2}$$
$$= 5.57 \times 10^{-48}\,\text{J}$$
$$= 3.47 \times 10^{-29}\,\text{eV}$$

2. The speed corresponding to this kinetic energy is:

$$v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2(5.57 \times 10^{-48}\,\text{J})}{10^{-9}\,\text{kg}}}$$
$$= 1.06 \times 10^{-19}\,\text{m/s}$$

Remarks: *We can see from this calculation that the minimum kinetic energy implied by the uncertainty principle is certainly not observable for macroscopic objects even as small as 10^{-6} g.*

EXAMPLE 5-8 An Electron in an Atomic Box **If the particle in a one-dimensional box of length $L = 0.1$ nm (about the diameter of an atom) is an electron, what will be its zero-point energy?**

Solution

Again using Equation 5-31, we find that

$$E \approx \frac{(\hbar c)^2}{2mc^2L^2} = \frac{(197.3\,\text{eV}\cdot\text{nm})^2}{2(0.511 \times 10^6\,\text{eV})(0.1\,\text{nm})^2} = 3.81\,\text{eV}$$

This is the correct order of magnitude for the kinetic energy of an electron in an atom.

Size of the Hydrogen Atom

The energy of an electron of momentum p a distance r from a proton is

$$E = \frac{p^2}{2m} - \frac{ke^2}{r}$$

If we take for the order of magnitude of the position uncertainty $\Delta x = r$, we have

$$(\Delta p)^2 = \overline{p^2} \geq \frac{\hbar^2}{r^2}$$

The energy is then

$$E = \frac{\hbar^2}{2mr^2} - \frac{ke^2}{r}$$

There is a radius r_m at which E is minimum. Setting $dE/dr = 0$ yields r_m and E_m:

$$r_m = \frac{\hbar^2}{ke^2m} = a_0 = 0.0529\,\text{nm}$$

and

$$E_m = -\frac{k^2e^4m}{2\hbar^2} = -13.6\,\text{eV}$$

The fact that r_m came out to be exactly the radius of the first Bohr orbit is due to the judicious choice of $\Delta x = r$ rather than $2r$ or $r/2$, which are just as reasonable. It should be clear, however, that any reasonable choice for Δx gives the correct order of magnitude of the size of an atom.

Widths of Spectral Lines

Equation 5-30 implies that the energy of a system cannot be measured exactly unless an infinite amount of time is available for the measurement. If an atom is in an excited state, it does not remain in that state indefinitely but makes transitions to lower energy states until it reaches the ground state. The decay of an excited state is a statistical process.

We can take the mean time for decay τ, called the ***lifetime***, to be a measure of the time available to determine the energy of the state. For atomic transitions, τ is of the order of 10^{-8} s. The uncertainty in the energy corresponding to this time is

$$\Delta E \geq \frac{\hbar}{\tau} = \frac{6.58 \times 10^{-16}\,\text{eV}\cdot\text{s}}{10^{-8}\,\text{s}} \approx 10^{-7}\,\text{eV}$$

This uncertainty in energy causes a spread $\Delta\lambda$ in the wavelength of the light emitted. For transitions to the ground state, which has a perfectly certain energy E_0 because of its infinite lifetime, the percentage spread in wavelength can be calculated from

$$E - E_0 = \frac{hc}{\lambda}$$

$$dE = -hc\frac{d\lambda}{\lambda^2}$$

$$|\Delta E| \approx hc\frac{|\Delta\lambda|}{\lambda^2}$$

thus

$$\frac{\Delta\lambda}{\lambda} \approx \frac{\Delta E}{E - E_0}$$

The energy width $\Gamma_0 = \hbar/\tau$ is called the ***natural line width***. Other effects that cause broadening of spectral lines are the Doppler effect, the recoil effect, and atomic collisions. For optical spectra in the eV energy range, the Doppler width D is about 10^{-6} eV at room temperature, i.e., roughly 10 times the natural width, and the recoil width is negligible. For nuclear transitions in the MeV range, both the Doppler width and the recoil width are of the order of eV, much larger than the natural line width. We shall see in Chapter 11 that in some special cases of atoms in solids at low temperatures, the Doppler and recoil widths are essentially zero and the width of the spectral line is just the natural width. This effect, called the ***Mössbauer effect*** after its discoverer, is extremely important, for it provides photons of well-defined energy, which are useful in experiments demanding extreme precision. For example, the 14.4-keV photon from ^{57}Fe has a natural width of the order of 10^{-11} of its energy.

Questions

7. What happens to the zero-point energy of a particle in a one-dimensional box as the length of the box $L \to \infty$?
8. Why is the uncertainty principle not apparent for macroscopic objects?

EXAMPLE 5-9 Emission of a Photon Most excited atomic states decay, i.e., emit a photon, within about $\tau = 10^{-8}$ s following excitation. What is the minimum uncertainty in the (1) energy and (2) frequency of the emitted photon?

Solution

1. The minimum energy uncertainty is the natural line width $\Gamma_0 = \hbar/\tau$; therefore,

$$\Gamma_0 = \frac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{2\pi \times 10^{-8}\,\text{s}} = \frac{4.14 \times 10^{-15}\,\text{eV}\cdot\text{s}}{2\pi \times 10^{-8}\,\text{s}} = 6.6 \times 10^{-8}\,\text{eV}$$

2. From de Broglie's relation $E = \hbar\omega$ we have

$$\Delta E = \hbar\Delta\omega = \hbar(2\pi\Delta f) = h\Delta f$$

so that Equation 5-30 can be written as

$$\Delta E\,\Delta t = h\Delta f\,\Delta t \geq \hbar$$

or

$$\Delta f\,\Delta t \geq \frac{1}{2\pi}$$

and the minimum uncertainty in the frequency becomes

$$\Delta f \geq \frac{1}{2\pi\Delta t} = \frac{1}{2\pi \times 10^{-8}}$$

$$\Delta f \geq 1.6 \times 10^{7}\,\text{Hz}$$

5-7 Wave-Particle Duality

We have seen that electrons, which were once thought of as simply particles, exhibit the wave properties of diffraction and interference. In earlier chapters we saw that light, which we previously had thought of as a wave, also has particle properties in its interaction with matter, as in the photoelectric effect or the Compton effect. All phenomena—electrons, atoms, light, sound—have both particle and wave characteristics. It is sometimes said that an electron, for example, behaves as both a wave and a particle. This may seem confusing since, in classical physics, the concepts of waves and particles are mutually exclusive. A *classical particle* behaves like a BB shot. It can be localized and scattered, it exchanges energy suddenly in a lump, and it obeys the laws of conservation of energy and momentum in collisions; but it does *not* exhibit interference and diffraction. A *classical wave* behaves like a water wave. It exhibits diffraction and interference patterns and has its energy spread out continuously in space and time, not quantized in lumps. Nothing, it was thought, could be both a classical particle and a classical wave.

We now see that the classical concepts do not adequately describe either waves or particles. Both matter and radiation have both particle and wave aspects. When emission or absorption is being studied, it is the particle aspects that are dominant. When matter or radiation propagates though space, wave aspects dominate. Notice that emission and absorption are events characterized by exchange of energy and discrete locations. For example, light strikes the retina of your eye and a photon is absorbed, transferring its energy to a particular rod or cone: an observation has occurred. This illustrates the point that *observations* of matter and radiation are described in terms of the particle aspects. On the other hand, predicting the intensity distribution of the light on your retina involves consideration of the amplitudes of waves that have propagated through space and been diffracted at the pupil. Thus, *predictions,* i.e., a priori statements about what may be observed, are described in terms of the wave aspects. Let's elaborate on this just a bit.

Every phenomenon is describable by a wave function that is the solution of a wave equation. The wave function for light is the electric field $\mathscr{E}(x, t)$ (in one dimension), which is the solution of a wave equation like Equation 5-11. We have called the wave function for an electron $\Psi(x, t)$. We shall study the wave equation of which Ψ is the solution, called the *Schrödinger equation,* in the next chapter. The magnitude squared of the wave function gives the probability (per unit volume) that the electron, if looked for, will be found in a given region. The wave function exhibits the classical wave properties of interference and diffraction. In order to predict where an electron, or other particle, is likely to be, we must find the wave function by methods similar to those of classical wave theory. When the electron (or light) interacts and exchanges energy and momentum, the wave function is changed by the interaction. The interaction can be described by classical particle theory, as is done in the Compton effect. There are times when classical particle theory and classical wave theory give the same results. If the wavelength is much smaller than any object or aperture, particle theory can be used as well as wave theory to describe wave propagation, because diffraction and interference effects are too small to be observed. Common examples are *geometric* optics, which is really a particle theory, and the motion of baseballs and jet aircraft. If one is interested only in time *averages* of energy and momentum exchange, the wave theory works as well as the particle theory. For example, the wave theory of light correctly predicts that the total electron current in the photoelectric effect is proportional to the intensity of the light.

More

That matter can exhibit wavelike characteristics as well as particlelike behavior can be a difficult concept to understand. A wonderfully clear discussion of wave-particle duality was given by R. P. Feynman and we have used it as the basis of our explanation on the home page of the *Two-Slit Interference Pattern* for electrons: www.whfreeman.com/modphysics4e See also Figures 5-22 and 5-23 and Equation 5-32 here.

Summary

TOPIC	RELEVANT EQUATIONS AND REMARKS	
1. De Broglie relations	$f = E/h$	5-1
	$\lambda = h/p$	5-2
	Electrons and all other particles exhibit the wave properties of interference and diffraction.	
2. Detecting electron waves		
Davisson and Germer	Showed that electron waves diffracted from a single Ni crystal according to Bragg's equation	
	$n\lambda = D \sin \phi$	5-5
3. Wave packets		
Wave equation	$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$	5-11
Uncertainty relations	$\Delta k\, \Delta x \approx 1$	5-16
	$\Delta\omega\, \Delta t \approx 1$	5-17
Wave speed	$v = \omega/k$	
Group (packet) speed	$v_g = d\omega/dk$	5-22
	$v_g = v_p + k\frac{dv_p}{dk}$	5-23
Matter waves	The wave packet moves with the particle speed, i.e., the particle speed is the group speed v_g.	
4. Probabilistic interpretation	The magnitude square of the wave function is proportional to the probability of observing a particle in the region dx at x and t.	
	$P(x)dx = \|\Psi\|^2 dx$	5-26
5. Heisenberg uncertainty principle	$\Delta x\, \Delta p \geq \frac{1}{2}\hbar$	5-29
	$\Delta E\, \Delta t \geq \frac{1}{2}\hbar$	5-30
	where each of the uncertainties is defined to be the standard deviation.	
Particle in a box	$\overline{E} \geq \frac{\hbar^2}{2mL^2}$	5-31
	The minimum energy of any particle in any "box" cannot be zero.	
Energy of H atom	The Heisenberg principle predicts $E_{min} = -13.6$ eV in agreement with the Bohr model.	

General References

The following general references are written at a level appropriate for the readers of this book.

De Broglie, L., *Matter and Light: The New Physics*, Dover, New York, 1939. In this collection of studies is de Broglie's lecture on the occasion of receiving the Nobel Prize, in which he describes his reasoning leading to the prediction of the wave nature of matter.

Feynman, R., "Probability and Uncertainty—The Quantum Mechanical View of Nature," filmed lecture, available from Educational Services, Inc., Film Library, Newton, Mass.

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Fowles, G. R., *Introduction to Modern Optics*, Holt, Rinehart & Winston, New York, 1968.

Hecht, E., *Optics*, 2d ed., Addison-Wesley, Reading, Mass., 1987.

Jenkins, F. A., and H. E. White, *Fundamentals of Optics*, 4th ed., McGraw-Hill, New York, 1976.

Mehra, J., and H. Rechenberg, *The Historical Development of Quantum Theory*, Vol. 1, Springer-Verlag, New York, 1982.

Resnick, R., and D. Halliday, *Basic Concepts in Relativity and Early Quantum Theory*, 2d ed., Wiley, New York, 1992.

Tipler, P., *Physics for Scientists and Engineers*, 4th ed., W. H. Freeman, New York, 1999. Chapters 15 and 16 include a complete discussion of classical waves.

Notes

1. Louis V. P. R. de Broglie (1892–1987), French physicist. Originally trained in history, he became interested in science after serving as a radio engineer in the French army (assigned to the Eiffel Tower) and through the work of his physicist brother Maurice. The subject of his doctoral dissertation received unusual attention because his professor, Paul Langevin (who discovered the principle on which sonar is based), brought it to the attention of Einstein, who described de Broglie's hypothesis to Lorentz as ". . . the first feeble ray of light to illuminate . . . the worst of our physical riddles." He received the Nobel Prize in physics in 1929, the first person so honored for work done for a doctoral thesis.

2. L. de Broglie, *New Perspectives in Physics* (New York: Basic Books, 1962).

3. See, e.g., P. Tipler, *Physics for Scientists and Engineers*, 4th ed. (New York: W. H. Freeman, 1999), Section 35-5.

4. Jean-Baptiste Perrin (1870–1942), French physicist. He was the first to show that cathode rays were actually charged particles, setting the stage for J. J. Thomson's measurement of their q/m ratio. He was also the first to measure the approximate size of atoms and molecules and determined Avogadro's number. He received the Nobel Prize in physics for that work in 1926.

5. Clinton J. Davisson (1881–1958), American physicist. He shared the 1937 Nobel Prize in physics with G. P. Thomson for demonstrating the diffraction of particles. Davisson's was the first ever awarded for work done somewhere other than at an academic institution. Germer was one of Davisson's assistants at Bell Telephone Laboratory.

6. Matter (electron) waves, like other waves, change their direction in passing from one medium (e.g., Ni crystal) into another (e.g., vacuum) in the manner described by Snell's law and the indices of refraction of the two media. For normal incidence Equation 5-5 is not affected, but for other incident angles it is altered a bit and that change has not been taken into account in either Figure 5-6 or 5-7.

7. *Nobel Prize Lectures: Physics* (Amsterdam and New York: Elsevier, 1964).

8. In spectroscopy, the quantity $k = \lambda^{-1}$ is called the *wave number*. In the theory of waves, the term *wave number* is used for $k = 2\pi/\lambda$.

9. If you are familiar with Fourier analysis, you will recognize that $y(x)$ and $A(k)$ are essentially Fourier transforms of each other.

10. This interpretation of $|\Psi|^2$ was first developed by the German physicist Max Born (1882–1970). One of his positions early in his career was at the University of Berlin, where he was to relieve Planck of his teaching duties. Born received the Nobel Prize in physics in 1954, in part for his interpretation of $|\Psi|^2$.

11. Werner K. Heisenberg (1901–1976), German physicist. After obtaining his Ph.D. under Sommerfeld, he served as an assistant to Born and to Bohr. He was the director of research for Germany's atomic bomb project during World War II. His work on quantum theory earned him the physics Nobel Prize in 1932.

12. The resolving power of a microscope is discussed in some detail in F. A. Jenkins and H. E. White, *Fundamentals of Optics*, 4th ed. (New York: McGraw-Hill, 1976), pp. 332–334. The expression for Δx used here is determined by Rayleigh's criterion that two points are just resolved if the central maximum of the diffraction pattern from one falls at the first minimum of the diffraction pattern of the other.

13. Richard P. Feynman (1918–1988), American physicist. This discussion is based on one in his classic text, *Lectures on Physics* (Reading, Mass.: Addison-Wesley, 1965). He shared the 1965 Nobel Prize in physics for his development of quantum electrodynamics (QED). It was Feynman who, while a member of the commission on the *Challenger* disaster, pointed out that the booster-stage O-rings were at fault. A genuine legend in American physics, he was also an accomplished bongo drummer and safecracker.

Problems

Level I

Section 5-1 The de Broglie Hypothesis

5-1. (*a*) What is the de Broglie wavelength of a 1-g mass moving at a speed of 1 m per year? (*b*) What should be the speed of such a mass if its de Broglie wavelength is to be 1 cm?

5-2. If the kinetic energy of a particle is much greater than its rest energy, the relativistic approximation $E \approx pc$ holds. Use this approximation to find the de Broglie wavelength of an electron of energy 100 MeV.

5-3. Electrons in an electron microscope are accelerated from rest through a potential difference V_0 so that their de Broglie wavelength is 0.04 nm. What is V_0?

5-4. Compute the de Broglie wavelengths of (*a*) an electron, (*b*) a proton, and (*c*) an alpha particle of 4.5 keV kinetic energy.

5-5. According to statistical mechanics, the average kinetic energy of a particle at temperature T is $3kT/2$, where k is the Boltzmann constant. What is the average de Broglie wavelength of nitrogen molecules at room temperature?

5-6. Find the de Broglie wavelength of a neutron of kinetic energy 0.02 eV (this is of the order of magnitude of kT at room temperature).

5-7. A free proton moves back and forth between rigid walls separated by a distance $L = 0.01$ nm. (*a*) If the proton is represented by a one-dimensional standing de Broglie wave with a node at each wall, show that the allowed values of the de Broglie wavelength are given by $\lambda = 2L/n$ where n is a positive integer. (*b*) Derive a general expression for the allowed kinetic energy of the proton and compute the values for $n = 1$ and 2.

5-8. What must be the kinetic energy of an electron if the ratio of its de Broglie wavelength to its Compton wavelength is (*a*) 10^2, (*b*) 0.2, and (*c*) 10^{-3}?

5-9. Compute the wavelength of a cosmic ray proton whose kinetic energy is (*a*) 2 GeV and (*b*) 200 GeV.

Section 5-2 Measurements of Particle Wavelengths

5-10. What is the Bragg scattering angle ϕ for electrons scattered from a nickel crystal if their energy is (*a*) 75 eV, (*b*) 100 eV?

5-11. Compute the kinetic energy of a proton whose de Broglie wavelength is 0.25 nm. If a beam of such protons is reflected from a calcite crystal with crystal plane spacing of 0.304 nm, at what angle will the first-order Bragg maximum occur?

5-12. (*a*) The scattering angle for 50-eV electrons from MgO is 55.6°. What is the crystal spacing D? (*b*) What would be the scattering angle for 100-eV electrons?

5-13. A certain crystal has a set of planes spaced 0.30 nm apart. A beam of neutrons strikes the crystal at normal incidence and the first maximum of the diffraction pattern occurs at $\phi = 42°$. What are the de Broglie wavelength and kinetic energy of the neutrons?

5-14. Show that in Davisson and Germer's experiment with 54-eV electrons using the $D = 0.215$ nm planes, diffraction peaks with $n = 2$ and higher are not possible.

5-15. A beam of electrons with kinetic energy 350 eV is incident normal to the surface of a KCl crystal which has been cut so that the spacing D between adjacent atoms in the planes parallel to the surface is 0.315 nm. Calculate the angle ϕ at which diffraction peaks will occur for all orders possible.

Section 5-3 Wave Packets

5-16. Information is transmitted along a cable in the form of short electric pulses at 100,000 pulses/s. (*a*) What is the longest duration of the pulses such that they do not overlap? (*b*)What is the range of frequencies to which the receiving equipment must respond for this duration?

5-17. Two harmonic waves travel simultaneously along a long wire. Their wave functions are $y_1 = 0.002 \cos(8.0x - 400t)$ and $y_2 = 0.002 \cos(7.6x - 380t)$, where y and x are in meters and t in seconds. (*a*) Write the wave function for the resultant wave in the form of Equation 5-15. (*b*) What is the phase velocity of the resultant wave? (*c*) What is the group velocity? (*d*) Calculate the range Δx between successive zeros of the group and relate it to Δk.

5-18. (*a*) Starting from Equation 5-23, show that the group velocity can also be expressed as $v_g = v_p - \lambda(dv_p/d\lambda)$. (*b*) The phase velocity of each wavelength of white light moving through ordinary glass is a function of the wavelength, i.e., glass is a dispersive medium. What is the general dependance of v_p on λ in glass? Is $dv_p/d\lambda$ positive or negative?

5-19. A radar transmitter used to measure the speed of pitched baseballs emits pulses of 2.0-cm wavelength that are 0.25 μs in duration. (*a*) What is the length of the wave packet produced? (*b*) To what frequency should the receiver be tuned? (*c*) What must be the minimum bandwidth of the receiver?

5-20. A certain standard tuning fork vibrates at 880 Hz. If the tuning fork is tapped, causing it to vibrate, then stopped a quarter of a second later, what is the approximate range of frequencies contained in the sound pulse that reached your ear?

5-21. If a phone line is capable of transmitting a range of frequencies $\Delta f = 5000$ Hz, what is the approximate duration of the shortest pulse that can be transmitted over the line?

5-22. (*a*) You are given the task of constructing a double-slit experiment for 5-eV electrons. If you wish the first minimum of the diffraction pattern to occur at 5°, what must be the separation of the slits? (*b*) How far from the slits must the detector plane be located if the first minima on each side of the central maximum are to be separated by 1 cm?

Section 5-4 The Probabilistic Interpretation of the Wave Function

5-23. A 100-g rigid sphere of radius 1 cm has a kinetic energy of 2 J and is confined to move in a force-free region between two rigid walls separated by 50 cm. (*a*) What is the probability of finding the center of the sphere exactly midway between the two walls? (*b*) What is the probability of finding the center of the sphere between the 24.9- and 25.1-cm marks?

5-24. A particle moving in one dimension between rigid walls separated by a distance L has the wave function $\Psi(x) = A \sin(\pi x/L)$. Since the particle must remain between the walls, what must be the value of A?

5-25. The wave function describing a state of an electron confined to move along the x axis is given at time zero by

$$\Psi(x, 0) = Ae^{-x^2/4\sigma^2}$$

Find the probability of finding the electron in a region dx centered at (*a*) $x = 0$, (*b*) $x = \sigma$, and (*c*) $x = 2\sigma$. (*d*) Where is the electron most likely to be found?

Section 5-5 The Uncertainty Principle

5-26. A tuning fork of frequency f_0 vibrates for a time Δt and sends out a waveform that looks like that in Figure 5-24. This wave function is similar to a harmonic wave except that it is confined to a time Δt and space $\Delta x = v\,\Delta t$, where v is the phase velocity. Let N be the approximate number of cycles of vibration. We can measure the frequency by counting the cycles and dividing by Δt. (*a*) The number of cycles is uncertain by approximately ±1 cycle. Explain why (see the figure). What uncertainty does this introduce in the determination of the frequency f? (*b*) Write an expression for the wave number k in terms of Δx and N. Show that the uncertainty in N of ±1 leads to an uncertainty in k of $\Delta k = 2\pi/\Delta x$.

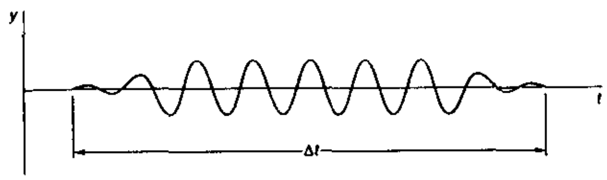

Fig. 5-24 Problem 5-26.

5-27. If an excited state of an atom is known to have a lifetime of 10^{-7} s, what is the uncertainty in the energy of photons emitted by such atoms in the spontaneous decay to the ground state?

5-28. A mass of 1 μg has a speed of 1 cm/s. If its speed is uncertain by 1 percent, what is the order of magnitude of the minimum uncertainty in its position?

5-29. ^{222}Rn decays by the emission of an α particle with a lifetime of 3.823 days. The kinetic energy of the α particle is measured to be 5.490 MeV. What is the uncertainty in this energy? Describe in one sentence how the finite lifetime of the excited state of the radon nucleus translates into an energy uncertainty for the emitted α particle.

5-30. If the uncertainty in the position of a wave packet representing the state of a quantum-system particle is equal to its de Broglie wavelength, how does the uncertainty in momentum compare with the value of the momentum of the particle?

5-31. In one of G. Gamow's Mr. Tompkins tales, the hero visits a "quantum jungle" where $\hbar$ is very large. Suppose that you are in such a place where $\hbar = 50$ J · s. A cheetah runs past you a few meters away. The cheetah is 2 m long from nose to tail tip and its mass is 30 kg. It is moving at 30 m/s. What is the uncertainty in the location of the "mid-point" of the cheetah? Describe in one sentence how the cheetah would look different to you than when $\hbar$ has its actual value.

5-32. In order to locate a particle, e.g., an electron, to within 5×10^{-12} m using electromagnetic waves ("light"), the wavelength must be at least this small. Calculate the momentum and energy of a photon with $\lambda = 5 \times 10^{-12}$ m. If the particle is an electron with $\Delta x = 5 \times 10^{-12}$ m, what is the corresponding uncertainty in its momentum?

5-33. The decay of excited states in atoms and nuclei often leaves the system in another, albeit lower-energy, excited state. (*a*) One example is the decay between two excited states of the nucleus of ^{48}Ti. The upper state has a lifetime of 1.4 ps, the lower state 3.0 ps. What is the fractional uncertainty $\Delta E/E$ in the energy of 1.3117-MeV gamma rays connecting the two states? (*b*) Another example is the H_α line of the hydrogen Balmer series. In this case the lifetime of both states is about the same, 10^{-8} s. What is the uncertainty in the energy of the H_α photon?

Section 5-6 Some Consequences of the Uncertainty Principle

5-34. A neutron has a kinetic energy of 10 MeV. What size object is necessary to observe neutron diffraction effects? Is there anything in nature of this size that could serve as a target to demonstrate the wave nature of 10-MeV neutrons?

5-35. The energy of a certain nuclear state can be measured with an uncertainty of 1 eV. What is the minimum lifetime of this state?

5-36. Show that the relation $\Delta p_s\,\Delta s > \hbar$ can be written $\Delta L\,\Delta\phi > \hbar$ for a particle moving in a circle about the z axis, where p_s is the linear momentum tangential to the circle, s is the arc length, and L is the angular momentum. How well can the angular position of the electron be specified in the Bohr atom?

5-37. An excited state of a certain nucleus has a half-life of 0.85 ns. Taking this to be the uncertainty Δt for emission of a photon, calculate the uncertainty in the frequency Δf, using Equation 5-28. If $\lambda = 0.01$ nm, find $\Delta f/f$.

Section 5-7 Wave-Particle Duality

There are no problems for this section.

Level II

5-38. A neutron in an atomic nucleus is bound to other neutrons and protons in the nucleus by the strong nuclear force when it comes within about 1 fm of another particle. What is the approximate kinetic energy of a neutron that is localized to within such a region? What would be the corresponding energy of an electron localized to within such a region?

5-39. Using the relativistic expression $E^2 = p^2c^2 + m^2c^4$, (*a*) show that the phase velocity of an electron wave is greater than *c*. (*b*) Show that the group velocity of an electron wave equals the particle velocity of the electron.

5-40. Show that if y_1 and y_2 are solutions of Equation 5-11, the function $y_3 = C_1y_1 + C_2y_2$ is also a solution for any values of the constants C_1 and C_2.

5-41. Show that if $\Delta x\,\Delta p = \frac{1}{2}\hbar$, the minimum energy of a simple harmonic oscillator is $\frac{1}{2}\hbar\omega = \frac{1}{2}hf$. What is the minimum energy in joules for a mass of 10^{-2} kg oscillating on a spring of force constant $K = 1$ N/m?

5-42. A particle of mass *m* moves in a one-dimensional box of length *L*. (Take the potential energy of the particle in the box to be zero so that its total energy is its kinetic energy $p^2/2m$.) Its energy is quantized by the standing-wave condition $n(\lambda/2) = L$, where λ is the de Broglie wavelength of the particle and *n* is an integer. (*a*) Show that the allowed energies are given by $E_n = n^2E_1$ where $E_1 = h^2/8mL^2$. (*b*) Evaluate E_n for an electron in a box of size $L = 0.1$ nm and make an energy-level diagram for the state from $n = 1$ to $n = 5$. Use Bohr's second postulate $f = \Delta E/h$ to calculate the wavelength of electromagnetic radiation emitted when the electron makes a transition from (*c*) $n = 2$ to $n = 1$, (*d*) $n = 3$ to $n = 2$, and (*e*) $n = 5$ to $n = 1$.

5-43. (*a*) Use the results of Problem 5-42 to find the energy of the ground state ($n = 1$) and the first two excited states of a proton in a one-dimensional box of length $L = 10^{-15}$ m $= 1$ fm. (These are of the order of magnitude of nuclear energies.) Calculate the wavelength of electromagnetic radiation emitted when the proton makes a transition from (*b*) $n = 2$ to $n = 1$, (*c*) $n = 3$ to $n = 2$, and (*d*) $n = 3$ to $n = 1$.

5-44. (*a*) Suppose that a particle of mass *m* is constrained to move in a one-dimensional space between two infinitely high barriers located *A* apart. Using the uncertainty principle, find an expression for the zero-point (minimum) energy of the particle. (*b*) Using your result from (*a*), compute the minimum energy of an electron in such a space if $A = 10^{-10}$ m and if $A = 1$ cm. (*c*) Calculate the minimum energy for a 100-mg bead moving on a thin wire between two stops located 2 cm apart.

5-45. A proton and a bullet each move with a speed of 500 m/s, measured with an uncertainty of 0.01 percent. If measurements of their respective positions are made simultaneous with the speed measurements, what is the minimum uncertainty possible in the position measurements?

Level III

5-46. Show that Equation 5-11 is satisfied by $y = f(\phi)$, where $\phi = x - vt$, for any function *f*.

5-47. An electron and a positron are moving toward each other with equal speeds of 3×10^6 m/s. The two particles annihilate each other and produce two photons of equal energy. (*a*) What were the de Broglie wavelengths of the electron and positron? Find the (*b*) energy, (*c*) momentum, and (*d*) wavelength of each photon.

5-48. It is possible for some fundamental particles to "violate" conservation of energy by creating and quickly reabsorbing another particle. For example, a proton can emit a π^+ according to $p \rightarrow n + \pi^+$ where the *n* represents a neutron. The π^+ has a mass of 140 MeV/c^2. The reabsorption must occur within a time Δt consistent with the uncertainty principle. (*a*) Considering that example by how much ΔE is energy conservation violated? (Ignore kinetic energy.) (*b*) For how long Δt can the π^+ exist? (*c*) Assuming that the π^+ is moving at nearly the speed of light, how far from the nucleus could it get in the time Δt? (As we will discuss in Chapter 11, this is the approximate range of the strong nuclear force.)

5-49. De Broglie developed Equation 5-2 initially for photons, assuming that they had a small but finite mass. His assumption was that RF waves with $\lambda = 30$ m traveled at a speed of at least 99 percent of that of visible light with $\lambda = 500$ nm. Beginning with the relativistic expression $hf = \gamma mc^2$, verify de Broglie's calculation that the upper limit of the rest mass of a photon is 10^{-44} g. (*Hint:* Find an expression for v/c in terms of hf and mc^2, and then let $mc^2 \ll hf$.) ($\gamma = 1/(1 - v^2/c^2)^{1/2}$.)

5-50. Suppose that you drop BBs onto a bull's-eye marked on the floor. According to the uncertainty principle, the BBs do not necessarily fall straight down from the release point to the center of the bull's-eye, but are affected by the initial conditions. (*a*) If the location of the release point is uncertain by an amount Δx perpendicular to the vertical direction and the horizontal component of the speed is uncertain by Δv_x, derive an expression for the minimum spread ΔX of impacts at the bull's-eye, if its is located a distance y_0 below the release point. (*b*) Modify your result in (*a*) to include the effect on ΔX of uncertainties Δy and Δv_y at the release point.

5-51. Using the first-order Doppler-shift formula $f' = f_0(1 + v/c)$, calculate the energy shift of a 1-eV photon emitted from an iron atom moving toward you with energy 3/2 kT at $T = 300$ K. Compare this Doppler line broadening with the natural line width calculated in Example 5-9. Repeat the calculation for a 1-MeV photon from a nuclear transition.

5-52. Calculate the order of magnitude of the shift in energy of a (*a*) 1-eV photon and (*b*) 1-MeV photon resulting from the recoil of an iron nucleus. Do this by first calculating the momentum of the photon, and then by calculating $p^2/2m$ for the nucleus using that value of momentum. Compare with the natural line width calculated in Example 5-9.