EM HW #4 - Physical Systems due 20 Feb 2007

3.12, 13, 23, 24

Problem 3.12 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0, and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$. See attached worksheet

Problem 3.13 For the infinite slot (Ex. 3.3) determine the charge density $\sigma(y)$ on the strip at $x = 0$, assuming it is a conductor at constant potential V_0.

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a). \qquad (3.36)$$

Recall from Ch. 2 that the E field changes across a charge distribution: $\Delta E = \frac{\sigma}{\epsilon_0} = \frac{-\partial V}{\partial n} = \frac{-\partial V}{\partial x}$ in this case

$$\frac{\partial V}{\partial x} = \frac{4V_0}{\pi} \sum \frac{1}{n} \underbrace{\frac{\partial}{\partial x} e^{-n\pi x/a}}_{-\frac{n\pi}{a} e^{-n\pi x/a}} \sin\frac{n\pi y}{a} \Big|_{x=0}$$

$$\frac{\partial V}{\partial x} = -\frac{4V_0}{a} \sum \underbrace{e^{-n\pi x/a}}_{\to 1 @ x=0} \sin\frac{n\pi y}{a} \Big|_{x=0}$$

$$\frac{\partial V}{\partial x} = -\frac{4V_0}{a} \sum \sin\frac{n\pi y}{a}$$

so $\sigma = -\epsilon_0 \frac{\partial V}{\partial x} = +\frac{4V_0 \epsilon_0}{a} \sum_{n=1,3,5} \sin\frac{n\pi y}{a}$

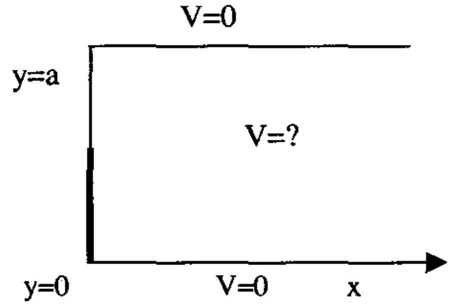
Separation of Variables: a technique for solving Laplace's equation $\nabla^2 V = 0$

Worksheet for fall E&M Problem 3.12 (p.136)

Find the potential in the infinite slot if the boundary at x=0 has two metal strips:

One, from from ($\frac{a}{2} < y \le a$), is at potential $-V_0$

The other, from ($0 \le y \le \frac{a}{2}$), is held at constant potential V_0



First, guess that Laplace's equation $\nabla^2 V = 0$ has solutions of the form

Just like Example 3.3

(1) V(x,y) = X(x) Y(y). If so, then the differential equation $\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$ (2)

becomes separable. Substitute (1) into (2) to get

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

(3)

Divide (3) by V = X Y and simplify:

$$\frac{\partial^2 X}{X \partial x^2} + \frac{\partial^2 Y}{Y \partial y^2} = 0$$

(4) I argued in class that each term must be constant

Find solutions to (5) $\frac{\partial^2 X}{\partial x^2} = k^2 X$ and (6) $\frac{\partial^2 Y}{\partial x^2} = -k^2 Y$

$X = X_0 e^{-kx}$ $\quad Y = C \sin ky + D \cos ky$ $\quad \frac{(3.27)}{129}$

(growing solution cannot fit BC that $V(x \to \infty) = 0$)

Substitute (5) and (6) into (1) for the general solution.

$V = X Y = e^{-kx} (C \sin ky + D \cos ky)$ (absorb X_0 into C & D)

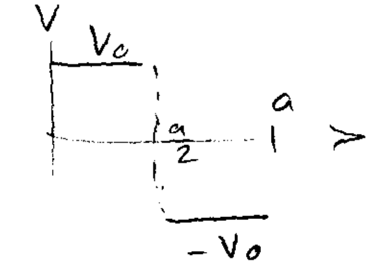
where $k = \frac{n\pi}{a}$ we may need both sin & cos to fit $V(x=0, y)$

Apply boundary conditions to find undetermined constants.

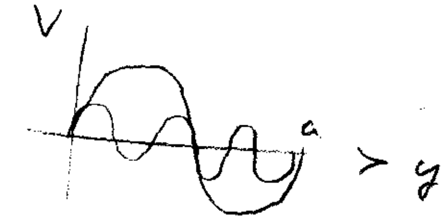
over

Now apply BC at $x=0$ to find C, as in Ex 3.3 p. 130:

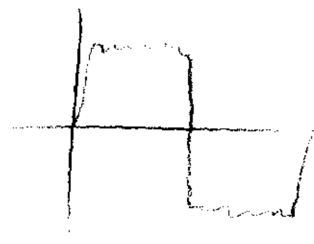
To match



we'll use the sine terms



since you can see these start to add up to something like our BC.



$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x} \sin \frac{n\pi}{a} y \qquad n=1,2,3,\ldots$$

where, by (3.34), $C_n = \frac{2}{a}\int_0^a V_0(y) \sin \frac{n\pi}{a} y \, dy$

$$C_n = \frac{2}{a}V_0 \int_0^{a/2} \sin \frac{n\pi}{a} y \, dy - \frac{2}{a}V_0 \int_{a/2}^{a} \sin \frac{n\pi}{a} y \, dy.$$ These are easy integrals:

$$= \frac{2}{a}V_0\left[-\frac{a}{n\pi}\cos \frac{n\pi}{a} y \Big|_0^{a/2}\right] - \frac{2}{a}V_0\left[-\frac{a}{n\pi}\cos \frac{n\pi}{a} y \Big|_{a/2}^{a}\right]$$

$$= \frac{2}{a}V_0\left\{-\frac{a}{n\pi}\left(\cos \frac{n\pi}{2} - \cos 0\right) + \frac{a}{n\pi}\left(\cos n\pi - \cos \frac{n\pi}{2}\right)\right\}$$

$$C_n = \frac{2V_0}{n\pi}\left\{-\cos \frac{n\pi}{2} + 1 + \cos n\pi - \cos \frac{n\pi}{2}\right\}$$

$$C_n = \frac{2V_0}{n\pi}\left\{1 + \cos n\pi - 2\cos \frac{n\pi}{2}\right\} = \frac{2V_0}{n\pi}\{\ \}$$

$n=1$: $\cos\pi = -1$ $\cos\frac{\pi}{2} = 0$: $\{\ \} = \{1-1+0\} = 0$ $\quad n=5\ldots$

$n=2$: $\cos 2\pi = 1$ $\cos\pi = -1$: $\{\ \} = \{1+1-2(-1)\} = 4$ $\quad n=6\ldots$

$n=3$: $\cos 3\pi = -1$ $\cos\frac{3\pi}{2} = 0$: $\{\ \} = \{1-1+0\} = 0$ $\quad n=7\ldots$

$n=4$: $\cos 4\pi = 1$ $\cos\frac{4\pi}{2} = 1$: $\{\ \} = \{1+1-2(1)\} = 0$ $\quad n=8\ldots$

$n=6$: $\cos 6\pi = +1$ $\cos 3\pi = -1$: $\{\ \} = \{1+1-2(-1)\} = 4$ $\quad n=10\ldots$

$C_n = 0$ if n is odd and if n is divisible by 4.

$C_n = \frac{2V_0}{n\pi}\cdot 4$ for $n = 2, 6, 10$, etc.

$$\boxed{V(x,y) = \frac{8V_0}{\pi}\sum \frac{1}{n} e^{-\frac{n\pi}{a}x} \sin \frac{n\pi}{a} y}$$

Separation of Variables: Laplace's equation $\nabla^2 V = 0$ in cylindrical coordinates

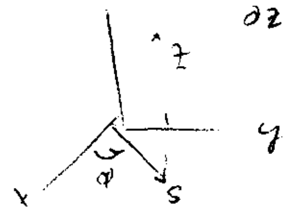
Worksheet for fall E&M Problem 3.23 (p.145)

Solve Laplace's eqn by separation of variables in cylindrical coordinates, assuming there is no dependence on z (this is cylindrical symmetry).

$$\frac{\partial V}{\partial z} = \frac{\partial S(s)\Phi(\varphi)}{\partial z} = 0$$

The Laplacian in cylindrical coordinates is eqn (1.82) p.44:

(1) $$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}$$



Look for solutions of the form $V(s,\phi) = S(s)\,\Phi(\phi)$ (2)

$$\frac{\partial V}{\partial s} = \Phi\frac{\partial S}{\partial s}, \quad \nabla^2 V = \Phi\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{s^2}S\frac{\partial^2\Phi}{\partial\varphi^2} + 0$$

Multiply by s^2 and divide by $V = S\,\Phi$:

$$\frac{\Phi}{S\Phi}\frac{s^2}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{s^2}{s^2}\frac{S}{S\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = \frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Both terms must be constant, and they must sum to zero, so the two constants are equal and opposite. Choose $-k^2$ for the Φ solution so it returns to its original value in one cycle.

Find solutions to (3) and (4)

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2 = C_1 \qquad\qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = C_2 = -k^2$$

(4) Since $\Phi(\phi + 2\pi) = \Phi(\phi)$, k must be an integer.

$$\frac{d^2\Phi}{d\varphi^2} = -k^2\Phi \longrightarrow \Phi = \Phi_0 e^{ik\varphi}$$

Show that $S = s^n$ is a solution to (3): what is the relation between n and k?

$$\frac{\partial S}{\partial s} = \frac{\partial}{\partial s}s^n = ns^{n-1}, \quad \frac{d}{ds}\left(s\left[ns^{n-1}\right]\right) = n\frac{\partial}{\partial s}s^n = n^2 s^{n-1}, \quad \frac{s}{S}\left[n^2 s^{n-1}\right] = k^2$$

$$\frac{1}{s^n}n^2 s^n = k^2$$

$$\boxed{n = \pm k}$$

Show that for k not 0, $S = A\,s^k + B\,s^{-k}$, and for $\boxed{k=0}$ $S = D + C \ln s$.

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0$$

$$\left(s\frac{dS}{ds}\right) = \text{const} \rightarrow \int dS = S = \int C\frac{ds}{s} = \underline{C\ln s + D} \quad \text{for } k=0$$

$$\underline{S = s^n = As^k + Bs^{-k}} \quad \text{for } k \neq 0$$

What is Φ for the k=0 case? Put it all together into a general solution.
Then apply it to problem 3.24.

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0 \; ; \; \frac{d\Phi}{d\varphi} = \text{constant} = a$$

no good: not periodic in φ.

$$\Phi = \int d\Phi = \int a\,d\varphi = a\varphi + b \longrightarrow \underline{\Phi = b} \quad k=0$$

$$\text{SOLUTION: } V = S\Phi = \underbrace{C\ln s + D}_{k=0} + \sum_{k=1}^{\infty}\left(As^k + Bs^{-k}\right)e^{\pm ik\varphi}$$

Problem 3.23 Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find *all* solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

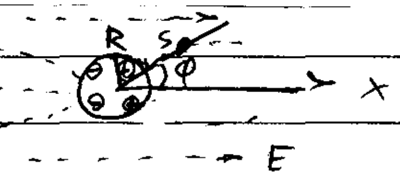
See attached worksheet. Solution:

$$V = \underbrace{C \ln s + D}_{k=0} + \sum_{k=1}^{\infty} \left(A s^{k} + B s^{-k}\right) e^{\pm i k \phi}$$

Problem 3.24 Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field $\mathbf{E_0}$. Find the surface charge induced on the pipe. [Use your result from Prob. 3.23.]

This is much like Ex 3.8, p. 141

The conducting pipe is an equipotential



Far from the pipe, $V \to -E_0 X + C$

(i) $V(s \gg R) = -E_0 s \cos\phi$

(ii) We can set $V(s=R) = 0$

We must fit these boundary conditions to the solution of the laplacian in cylindrical coordinates, above:

$$V = C \ln s + D + \sum_{k=1}^{\infty} \left(A s^{k} + B s^{-k}\right)\left(c \cos k\phi - d \sin k\phi\right)$$

C and $D = 0$ because $V = 0$ at $s = R$ – no constant terms

$d = 0$ because of the orientation of the E field – no sines

(i) And the only wave number is $k = 1$, since $V \sim \cos\phi$

so $V = \left(a s + \frac{b}{s}\right)\cos\phi$ where I combined $a = \frac{A}{c}$, $b = Bc$

BC (ii) $V(s=R) = 0 = \left(aR + \frac{b}{R}\right)\cos\phi \to aR^2 = -b$

BC (i) $V(s \gg R) = \left(as + \frac{b}{\infty}\right)\cos\phi \simeq a s \cos\phi = -E_0 s \cos\phi$

$a = -E_0 \to b = +E_0 R^2$

$$V = \left(as + \frac{b}{s}\right)\cos\phi = \left(-E_0 s + \frac{E_0 R^2}{s}\right)\cos\phi$$

$$V = E_0 s\left(-1 + \frac{R^2}{s^2}\right)\cos\phi$$

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial s}\Big|_{s=R} = -\epsilon_0 E_0 \cos\phi \frac{\partial}{\partial s}\left[s\left(-1 + \frac{R^2}{s^2}\right)\right]$$

$$\frac{\partial}{\partial s}\left[-s + \frac{R^2}{s}\right] = -1 - \frac{R^2}{s^2}$$

$$\sigma = +\epsilon_0 E_0 \cos\phi \left(1 + \frac{R^2}{s^2}\right)\Big|_{s=R} = \epsilon_0 E_0 \cos\phi\,(1+1)$$

$$\sigma = 2\epsilon_0 E_0 \cos\phi$$