Modern Physics HW due week 4 Thus 1 Feb

Ch 3 # 15, 23, 58, 25

Ch 4 # 3, 17, 24

E1 Z7n

3-15. As noted in the chapter, the cosmic microwave background radiation fits the Planck equations for a blackbody at 2.7 K. (*a*) What is the wavelength at the maximum intensity of the spectrum of the background radiation? (*b*) What is the frequency of the radiation at the maximum? (*c*) What is the total power incident on Earth from the background radiation?

$$\lambda_p T = 3\times10^{-3}\ mK$$

$$\lambda_p = \frac{3\times10^{-3}\ mK}{2.7K} \sim 10^{-3}\ m$$

(b) $E = \frac{hc}{\lambda} = hf \rightarrow f = \frac{c}{\lambda} = \frac{3\times10^{8}\ \frac{m}{s}}{10^{-3}\ m} = 3\times10^{11}\ Hz$

(c) $\frac{\text{Power}}{\text{area}} = \sigma T^4$ is incident on an effective area of a flat DISK of radius $= R_{earth}$, due to projection effects (more at equator, less at poles)

OOPS - That would be for Sun but for a source from all directions. $A = 4\pi R^2$ after all - SPHERE.

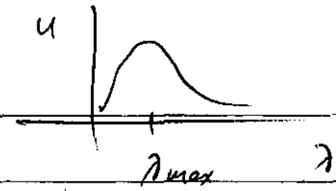
$$\text{Power} = \sigma T^4 \cdot \text{area} = \sigma T^4 4\pi R^2 =$$

$$= 4\pi \left(5.67\times10^{-8}\ \frac{W}{m^2K^4}\right)\left(6.4\times10^{6}\ m\right)^2 (2.7K)^4$$

$$= 1.5\times10^{9}\ W$$

3.23. Use Planck's Law $u(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/\lambda kT} - 1}$ $\frac{(3.33)}{p.138}$ to derive the Constant in Wien's Law

$\lambda_{peak} T = 3 \times 10^{-3}$ m·K $\frac{(3.20)}{p.134}$



Let $u = \frac{A\lambda^{-5}}{e^{B/\lambda} - 1}$. We want to find where u peaks, or where $\frac{\partial u}{\partial \lambda} = 0$. Solve for λ.

$$\frac{\partial u}{\partial \lambda} = A\left[\lambda^{-5}\frac{\partial}{\partial\lambda}(e^{B/\lambda}-1)^{-1} + (e^{B/\lambda}-1)^{-1}\frac{\partial}{\partial\lambda}\lambda^{-5}\right]$$

$$= A\left[\frac{1}{\lambda^5}\frac{(-1)}{(e^{B/\lambda}-1)^2}\frac{\partial}{\partial\lambda}(e^{B/\lambda}-1) + \frac{1}{(e^{B/\lambda}-1)}\left(\frac{-5}{\lambda^6}\right)\right]$$

$$= \frac{-A}{\lambda^5(e^{B/\lambda}-1)}\left[\frac{e^{B/\lambda}\frac{\partial}{\partial\lambda}(B\lambda^{-1})}{(e^{B/\lambda}-1)} + \frac{5}{\lambda}\right]$$

$$\frac{du}{d\lambda} = \frac{-A}{\lambda^5(e^{B/\lambda}-1)}\left[\frac{e^{B/\lambda}(-B\lambda^{-2})}{(e^{B/\lambda}-1)} + \frac{5}{\lambda}\right]$$

$$= 0 \text{ when } \frac{e^{B/\lambda}B}{(e^{B/\lambda}-1)\lambda^2} = \frac{5}{\lambda}$$

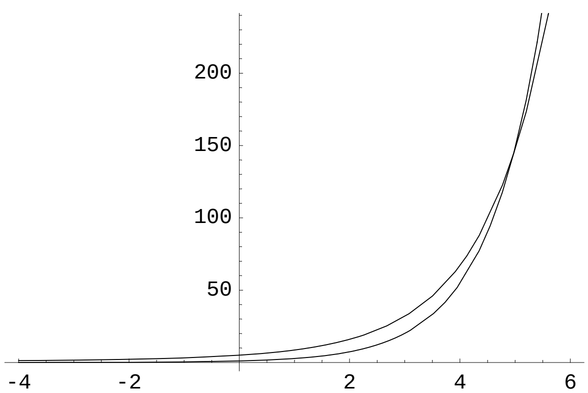
$$\text{or} \quad e^{B/\lambda}B = 5\lambda(e^{B/\lambda}-1)$$

$$B = 5\lambda(1 - e^{-B/\lambda})$$

This can be solved graphically or numerically!

Planck ' s Blackbody equation –
Show that it reduces to Wien ' s Law. (EJZ Feb 2006 Physics of Astronomy)
Solve graphically the equation derived for Raff 11.9 :
Planck ' s equation has an extremum when
b * Exp[b / x] = 5 x (Exp[b / x] – 1) where x = lambda. Write in terms of y = b / x.

```
In[3]:= Clear[f, g];
        f[y_] := Exp[y];
        g[y_] := (5 / y) (Exp[y] - 1)

In[6]:= f[y]

Out[6]= ⅇ^y

In[7]:= g[y]

Out[7]= 5 (-1 + ⅇ^y)
        ------------
             y

In[24]:= Clear[p1, p2];
         p1 = Plot[f[y], {y, -4, 6}];
         p2 = Plot[g[y], {y, -4, 6}];

In[20]:= Show[p1, p2]
```



Looks like these functions cross at about y=5.

```
In[22]:= NSolve[f[y] == g[y], y]

Out[22]= {{y → 4.96511}}
```

Solution: Planck's equation has an extremum when f=g when y=4.965

See attached 4.965

Graphical solution: $Be^{B/\lambda} = 5\lambda(e^{B/\lambda}-1)$ when $\frac{B}{\lambda} \approx 5$

$$BT = \frac{hc}{k} = \frac{1240\ \text{eV}\cdot\text{nm}}{1.38\times10^{-23}\frac{J}{K}\left|\frac{\text{eV}}{1.6\times10^{-19}J}\right|} = 8.63\times10^{-5}\ \text{eV/K}$$

$$BT = 1.44\times10^{7}\ K\cdot\text{nm}\left|\frac{m}{10^{9}\ \text{nm}}\right| = 1.44\times10^{-2}\ K\cdot m$$

$$B = 5\lambda = \frac{1.44\times10^{-2}\ K\cdot m}{T}$$

$$\lambda T = \frac{1.44\times10^{-2}\ K\cdot m}{5} = 2.9\times10^{-3}\ m\cdot K$$

Wien's constant derived from Planck's blackb

3-25. The orbiting space shuttle moves around Earth well above 99 percent of the atmosphere, yet it still accumulates an electric charge on its skin due, in part, to the loss of electrons caused by the photoelectric effect with sunlight. Suppose the skin of the shuttle is coated with Ni, which has a relatively large work function $\phi = 4.87$ eV at the temperatures encountered in orbit. (*a*) What is the maximum wavelength in the solar spectrum that can result in the emission of photoelectrons from the shuttle's skin? (*b*) What is the maximum fraction of the total power falling on the shuttle that could potentially produce photoelectrons?

Photoelectric effect $\quad$ Energy$_{in}$ = Energy$_{out}$

$$\text{Light} = hf = \underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic}} + \underbrace{\phi}_{\text{work function}}$$

If you turn on a repelling potential V_0, it lowers the maximum kinetic energy of emitted particles:

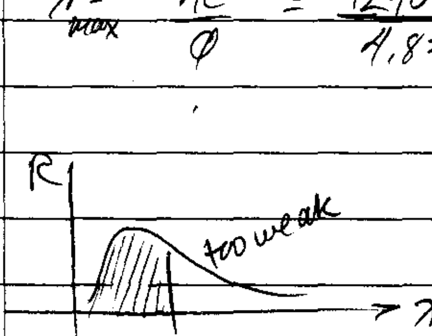
$$eV_0 = \left(\tfrac{1}{2}mv^2\right)_{max} = hf - \phi$$

(a) With no stopping potential, consider barely ejected electrons with $KE \to 0$. Then $hf = \frac{hc}{\lambda} \to \phi$

$$\lambda_{max} = \frac{hc}{\phi} = \frac{1240\ \text{eV}\cdot\text{nm}}{4.87\ \text{eV}} = 255\ \text{nm}$$

– this is UV, so less energetic (e.g. optical) photons don't contribute.

(b) Sun's $\frac{\text{Power}}{\text{area}}$ curve



If λ_{max} is the largest wavelength that can photo-eject electrons,

then $f = \dfrac{\text{Power in sufficiently energetic photons}}{\text{total incident power}} = \dfrac{R(\text{shaded area})}{R(\text{total area under curve})}$

total area $= R = \sigma T^4 = \left(5.67\times10^{-8}\ \frac{W}{m^2K^4}\right)(5800\ K)^4 = 6.4\times10^7\ W/m^2$

Shaded area $\approx u(\lambda)\Delta\lambda$ where $\Delta\lambda \sim \lambda_{max}$ and $u(\lambda)$

$$\lambda = \frac{\lambda_{max}}{2} = 128\ \text{nm}$$

Note that this is a rough estimate, so don't bother with high precision!

$$u(\lambda) = \frac{8\pi hc}{\lambda^5 (e^{hc/\lambda kT} - 1)}$$

$hc = 1240 \text{ eV}\cdot\text{nm} = 2\times10^{-25} \text{ J}\cdot$

$kT = 8.63\times10^{-5} \frac{\text{eV}}{\text{K}} \cdot 5800\text{K} = 0.5 \text{ e}$

$$\frac{hc}{\lambda(kT)} = \frac{1240 \text{ eV}\cdot\text{nm}}{128 \text{ nm} (0.5 \text{ eV})} = 19.4$$

$$\frac{hc}{\lambda^5} = \frac{2\times10^{-25} \text{ J}\cdot\text{m}}{(128\times10^{-9}\text{m})^5} = 5.8\times10^{-9} \text{ J/m}^4$$

$$u(\lambda) = \frac{8\pi (5.8\times10^{-9} \text{ J/m}^4)}{(e^{19.4} - 1)} = 549 \frac{\text{J}}{\text{m}^4}$$

$$u(\lambda)\Delta\lambda \approx u(\lambda)\lambda_{max} = 549 \frac{\text{J}}{\text{m}^4} \times 255\times10^{-9} \text{ m} = 1.4\cdot10^{-4}$$

power in sufficiently energetic photons =

$$R(\text{shaded}) = \frac{c}{4} u(\lambda) \lambda_{max} = \frac{3\times10^8 \frac{\text{m}}{\text{s}}}{4} \times 1.4\times10^{-4} \frac{\text{J}}{\text{m}^3} = 1\times10^4 \text{ W/}$$

total incident power $= R = 6.4\times10^7 \text{ W/m}^2$

$$\text{fraction of power} = \frac{R\text{shaded}}{R} = \frac{10^4 \text{ W/m}^2}{6.4\times10^7 \text{ W/m}^2} = 1.6\times10^{-4}$$

can potentially produce photoelectrons

3-58, Derive (3.32) from (3.30) and (3.31) p.138

$$\sum_{n=0}^{\infty} f_n = A \sum_{n=0}^{\infty} e^{-n\epsilon/kT} = 1 \qquad (3\text{-}30)$$

The average energy of an oscillator is then given by the discrete-sum equiv Equation 3-27,

$$\overline{E} = \sum_{n=0}^{\infty} E_n f_n = \sum_{n=0}^{\infty} E_n A e^{-E_n/kT} \qquad (3\text{-}31)$$

Calculating the sums in Equations 3-30 and 3-31 (see Problem 3-58) yields the

$$\overline{E} = \frac{\epsilon}{e^{\epsilon/kT} - 1} = \frac{hf}{e^{hf/kT} - 1} = \frac{hc/\lambda}{e^{hc/\lambda kT} - 1} \qquad (3\text{-}32)$$

Multiplying this result by the number of oscillators per unit volume in the inte given by Equation 3-23, we obtain for the energy density distribution functior radiation in the cavity: $(3\text{-}33)\ u(\lambda) = \dfrac{8\pi hc\,\lambda^{-5}}{e^{hc/\lambda kT} - 1}$

(3-23) p.136 $u(\lambda) = \dfrac{8\pi}{\lambda^4}$

$(3\text{-}29)\quad f_n = Ae^{-E_n/kT} = Ae^{-n\epsilon/kT}$

$E_n = n\epsilon = nhf$ where n = integer

ϵ = quantized energy of each oscillator

f = quantized frequency

h = Planck constant

Let $e^{-n\epsilon/kT} = e^{-nhf/kT} = e^{-nx}$ $(x = hf/kT)$

Then $\displaystyle\sum_{n=0}^{\infty} f_n = A\sum_{n=0}^{\infty} e^{-nx} = A[e^0 + e^{-x} + (e^{-x})^2 + \ldots]$

$= A[1 + y + y^2 + \ldots] \equiv 1$ by definition

$(y = e^{-x})$

Trick: Compare this to the power series expansion for $(1-y)^{-1}$

$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \ldots$

$(1-y)^{-1} = 1 - (-y) + \frac{-1(-2)}{2!}(-y)^2 + \frac{(-1)(-2)(-3)}{3!}(-y)^3 = 1 + y + \frac{2y^2}{2} + \ldots$

$$= A \sum e^{-nx}$$

Then $\sum f_n = A(1-y)^{-1} \equiv 1 \rightarrow A = \underline{(1-y)}$

$$(3\text{-}31)\ \bar{E} = \sum_{n=0}^{\infty} E_n A e^{-E_n/kT} = A \sum_{n=0}^{\infty} \underbrace{(nhf)}_{E_n} e^{-nhf/kT} = Ahf \sum_{n=0}^{\infty} n e^{-nx}$$

Trick: $\frac{d}{dx} e^{-nx} = \underline{-n e^{-nx}}$ and we found $\sum e^{-nx} = (1-y)^{-1}$

Combine these to get $\sum n e^{-nx} = -\frac{d}{dx} \sum e^{-nx} = -\frac{d}{dx}\left((1-y)^{-1}\right)$

Evaluate in general and simplify

$$-\frac{d}{dx}\left((1-y)^{-1}\right) = -\frac{dy}{dx}\frac{d}{dy}(1-y)^{-1} = -\frac{dy}{dx}(-1)(1-y)^{-2}\frac{d}{dy}(1-y) = \frac{dy}{dx}\frac{-1}{(1-y)^2}$$

Then use $\frac{dy}{dx} = \frac{d}{dx}(e^{-x}) = \underline{-e^{-x}} \equiv -y$

Sub into (3-31) and show that $\bar{E} = \frac{hfy}{1-y}$:

$$\sum n e^{-nx} = -\frac{d}{dx}(1-y)^{-1} = \frac{dy}{dx}\frac{-1}{(1-y)^2} = +y/(1-y)^2$$

$$\bar{E} = Ahf \sum n e^{-nx} = (1-y)\frac{hfy}{(1-y)^2} = \frac{hfy}{(1-y)}$$

Sub in $y = e^{-x}$, multiply by $\frac{e^{+x}}{e^{+x}}$, Substitute in $x = hf/kT$ to get result:

$$\bar{E} = \frac{hf e^{-x}}{(1-e^{-x})}\left(\frac{e^{x}}{e^{x}}\right) = \frac{hf}{e^{x}-1} = \frac{hf}{e^{hf/kT}-1} = \frac{hc/\lambda}{e^{hc/kT\lambda}-1}$$ ✓

Modern Ch 4 #3, 17, 24

4-3. An astronomer finds a new absorption line with λ = 164.1 nm in the ultraviolet region of the sun's continuous spectrum. He attributes the line to hydrogen's Lyman series. Is he right? Justify your answer.

$m = 1$ for Lyman Series

$\Delta E = \frac{hc}{\lambda} = E_m - E_n$ where $E_n = \frac{E_1}{n^2}$, $E_1 = -13.6\,eV$

$\frac{hc}{\lambda} = E_1 - \frac{E_1}{n^2} = E_1\left(1 - \frac{1}{n^2}\right)$

λ?

Is there some integer n that will give the observed

$\left(1 - \frac{1}{n^2}\right) = \frac{hc}{E_1 \lambda} = \frac{1240\,eV \cdot nm}{13.6\,eV \cdot 164.1\,nm} = 0.5556$

$\frac{1}{n^2} = 1 - 0.5556 = 0.4444$

$n = (0.4444)^{-1/2} = 1.5$ - NOT AN INTEGER

$\therefore$ Not a Lyman line.

4-17. Light of wavelength 410.7 nm is observed in emission from a hydrogen source. (a) What transition between hydrogen Bohr orbits is responsible for this radiation? (b) To what series does this transition belong?

$\lambda = 410.7\,nm$ is visible, so

this is a Balmer series line, a transition to $n = 2$.

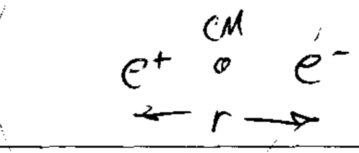
$\frac{hc}{\lambda} = \frac{E_1}{2^2} - \frac{E_1}{n^2} = E_1\left(\frac{1}{4} - \frac{1}{n^2}\right) \rightarrow \frac{hc}{\lambda E_1} = \frac{1}{4} - \frac{1}{n^2}$

$\frac{1}{n^2} = \frac{1}{4} - \frac{hc}{\lambda E_1} = \frac{1}{4} - \frac{1240\,eV \cdot nm}{410.7\,nm \cdot 13.6\,eV} = 2.8 \times 10^{-2}$

$n = (2.8 \times 10^{-2})^{-1/2} = 6$

POSITRONIUM MUTUAL ORBIT p. 98

4-34. The electron-positron pair that was discussed in Chapter 2 can form a hydrogenlike system called *positronium*. Calculate (*a*) the energies of the three lowest states and (*b*) the wavelength of the Lyman α and β lines. (Detection of those lines is a "signature" of positronium formation.)



$$\mu = \frac{m \cdot m}{m+m} = \frac{m}{2}$$

The easy way: just use $E_n = -hc\frac{R}{n^2}$ (4-23)
and the reduced mass correction
to the Rydberg constant: $R = R_\infty\left(\frac{1}{1+\frac{m}{M}}\right) = \frac{R_\infty}{2}$

Or, a real derivation of the energy levels, using the reduced mass $\mu = \frac{m \cdot m}{m+m} =$ (As done in class!)

$$\frac{1}{2}\mu v^2 = \frac{kqQ}{r} = \frac{ke^2}{r} \qquad L = n\hbar = \mu v r$$

$$v^2 = \frac{2ke^2}{\mu r} \qquad v^2 = \left(\frac{n\hbar}{\mu r}\right)^2$$

Eliminate v^2 and solve for r

$$\frac{2ke^2}{\mu r} = \frac{n^2\hbar^2}{\mu^2 r^2} \rightarrow r = \frac{n^2\hbar^2}{\mu\, 2ke^2}$$

Substitute r into (virial theorem!) $E_{tot} = \frac{1}{2}V = -\frac{1}{2}\frac{ke^2}{r}$

$$E_{tot} = E_n = -\frac{1}{2}ke^2\left(\frac{\mu\, 2ke^2}{n^2\hbar^2}\right) = -\frac{k^2e^4\mu}{n^2\hbar^2}$$

Compare to Bohr energy: $E = -\frac{k^2e^4 m}{n^2\hbar^2}$ where $m = 2\mu$

Either way, you should get half the energy of H atom: $E_0 = -13.6$ eV

(a) $E_1 = E_0/2 = {}^{-13.6\,eV}/_2 = -6.8\ eV$ = ground state of positronium

$E_2 = \frac{E_1}{2^2} = \frac{-6.8\,eV}{4} = \underline{-1.7}\ eV$

$E_3 = \frac{E_1}{3^2} = \frac{-6.8\,eV}{9} = \underline{-0.76}\ eV$

(b) Lyman series lines are from transitions to the $n=1$ level.

Lyman α: $E_2 \to E_1$ Lyman β: $E_3 \to E_1$

$E_2 - E_1 = \underline{5.1}\ eV$ $E_3 - E_1 = \underline{6.04}\ eV$

$-1.7+6.8$ $-0.76+6.8$

$$\Delta E = \frac{hc}{\lambda}$$

$\lambda_{21} = \frac{1240\,eV\cdot nm}{5.1\,eV}$ $\lambda_{31} = \frac{1240\ eV\cdot nm}{6.04\,eV}$

$\lambda_{21} = 243\ nm$ $\lambda_{31} = 205\ nm$