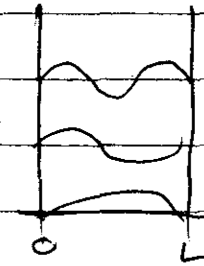
Modern Physics HW #5 due 1 Mar 2007 E/Z

① Expectation values in square well (from lecture) ② Griffiths QM 2.13
③ 2.14

① Expectation values



Exercise: Consider the infinite square well of width L.

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

(a) What is <x>? $= \int_{-\infty}^{\infty} x \psi^2 dx$ — **A: L/2**

(b) What is <x²>? $= \int_{-\infty}^{\infty} x^2 \psi^2 dx$ — $B: \langle x^2 \rangle = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$

(c) What is <p>? (Guess first) — **C: <p>=0**

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \left(\psi^* \frac{\partial \psi}{\partial x}\right) dx$$

(d) What is <p²>? (Guess first) — **D: <p²>=2mE**

ⓐ By symmetry, $\langle x \rangle = \frac{L}{2}$. Calculating it,

$$\langle x \rangle = \frac{2}{L}\int_0^L x \sin^2 \frac{n\pi x}{L} dx \quad \text{Let } y = \frac{n\pi x}{L}, \quad dy = \frac{n\pi}{L} dx$$

$$x = \frac{Ly}{n\pi} \qquad \frac{L}{n\pi} dy = dx$$

(Limit $x = L$: $y = n\pi$

$$\langle x \rangle = \frac{2}{L}\int \left(\frac{Ly}{n\pi}\right) \sin^2 y \left(\frac{L}{n\pi}\right) dy$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \int y \sin^2 y \, dy. \quad \text{Schaum (14.348) p. 76}$$

$$\int_0^{n\pi} y \sin^2 y \, dy = \frac{y^2}{4} - \frac{y \sin 2y}{4} - \frac{\cos 2y}{8} \Bigg|_0^{n\pi}$$

$$= \frac{1}{4}(n\pi)^2 - \frac{1}{4}(n\pi \sin 2n\pi) - \frac{1}{8}(\cos 2n\pi - \cos 0)$$

$$= \frac{1}{4}(n\pi)^2 - 0 - 0$$

$$\langle x \rangle = \frac{2L}{(n\pi)^2} \frac{1}{4}(n\pi)^2 = \frac{L}{2}$$

as predicted by symmetry.

① infinite square well...

$$y = \frac{n\pi x}{L}$$

ⓑ $\langle x^2 \rangle = \int_0^L x^2 \frac{2}{L} \sin^2 \frac{n\pi x}{L}\, dx = \frac{2}{L} \int_0^{n\pi} \left(\frac{L}{n\pi}\right)^3 y^2 \sin^2 y\, dy$

Dwight $\frac{430,22}{94}$ $\int_0^{n\pi} y^2 \sin^2 y\, dy = \frac{y^3}{6} - \left(\frac{y^2}{4} - \frac{1}{8}\right) \sin 2y - \frac{y \cos 2y}{4} \Big|_0^{n\pi}$

$$= \frac{(n\pi)^3}{6} - \frac{1}{4}\left((n\pi)^2 - \frac{1}{2}\right)\left(\sin 2n\pi^{\,\to 0} - \sin 0\right) - \frac{n\pi}{4}\left(\cos 2n\pi - \cos 0\right)$$

$$= \frac{(n\pi)^3}{6} - 0 - \frac{n\pi}{4}(1)$$

$$\langle x^2 \rangle = \frac{2L^2}{(n\pi)^3} \left[\frac{(n\pi)^3}{6} - \frac{n\pi}{4}\right]$$

$$\langle x^2 \rangle = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$$

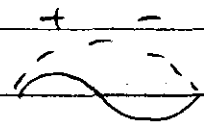
1ⓒ $\langle p \rangle = 0$ by symmetry: just as likely to be moving left as right. Calculate it:

$$\langle p \rangle = -i\hbar \int_0^L \psi \frac{\partial \psi}{\partial x}\,dx \qquad \psi = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

$$\frac{\partial \psi}{\partial x} = \sqrt{\frac{2}{L}}\;\frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right)$$

$$\int \psi \frac{d\psi}{dx}\,dx = \frac{2}{L}\,\frac{n\pi}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

ODD EVEN



sin × cos over one period = 0

Alternate calculation:

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} \quad \text{and} \quad \langle x \rangle = L/2 \quad \text{so} \quad \frac{d\langle x \rangle}{dt} = \frac{d\,L/2}{dt} = 0$$

1ⓓ $\langle E \rangle = \dfrac{\langle p^2 \rangle}{2m} = E_n$ so $\underline{\langle p^2 \rangle = 2mE_n}$

I will need A_0 for QHO

$$\int_{-\infty}^{\infty} \psi_0^2\, dx = A_0^2 \int e^{-2ax^2} dx \quad \text{where } a = m\omega/2\hbar$$

let $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$, then $\xi^2 = \frac{2m\omega}{2\hbar} x^2 = 2ax^2$

$$d\xi = \sqrt{\frac{m\omega}{\hbar}}\, dx \quad , \quad dx = \sqrt{\frac{\hbar}{m\omega}}\, d\xi$$

$$\int_{-\infty}^{\infty} \psi_0^2\, dx = A_0^2 \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi$$

GAUSSIAN INTEGRALS: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (1)

a : integer > 0 $\quad \int_{-\infty}^{\infty} x^{(2a-1)} e^{-x^2} dx = 0 \quad (x^{odd})$

$(x^{even}) \quad \int_{-\infty}^{\infty} x^{2a} e^{-x^2} dx = \frac{1.3.5\ldots(2a-1)\sqrt{\pi}}{2^a}$

$$\int_{-\infty}^{\infty} \psi_0^2\, dx = A_0^2 \sqrt{\frac{\hbar}{m\omega}}\, \sqrt{\pi} = 1$$

$$A_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

3.5

*Problem (2.13) Using the methods and results of this section, QHO

(a) Normalize ψ_1 (Equation 2.51) by direct integration. Check your answer a[...] the general formula (Equation 2.54). *Note*: In this and most problems inv[...] the harmonic oscillator, it simplifies the notation if you introduce the va[...] $\xi \equiv \sqrt{m\omega/\hbar}\,x$ and the constant $\alpha \equiv (m\omega/\pi\hbar)^{1/4}$.

(b) Find ψ_2, but don't bother to normalize it.

(c) Sketch ψ_0, ψ_1, and ψ_2.

(d) Check the orthogonality of ψ_0, ψ_1, and ψ_2. *Note*: If you exploit the eve[...] and oddness of the functions, there is really only one integral left to ev[...] explicitly.

$$\psi_0(x) = A_0 e^{-m\omega x^2/2\hbar}$$

$$\psi_1(x) = A_1\sqrt{\frac{m\omega}{\hbar}}\,x e^{-m\omega x^2/2\hbar}$$

$$\psi_2(x) = A_2\left(1 - \frac{2m\omega x^2}{\hbar}\right)e^{-m\omega x^2/2\hbar}$$

$x^2 = \frac{\hbar}{m\omega}\xi^2$

(a) $\displaystyle\int_{-\infty}^{\infty} \psi_1^2\,dx = 1 = A_1^2\,\frac{m\omega}{\hbar}\int x^2 e^{-\frac{m\omega x^2}{\hbar}}\,dx = \frac{m\omega}{\hbar}\sqrt{\frac{\hbar}{m\omega}}\left(\frac{\hbar}{m\omega}\right)\int \xi^2 e^{-\xi^2}\,d\xi$

$dx = \sqrt{\frac{\hbar}{m\omega}}\,d\xi$

Gaussian Integral: $\displaystyle\int_{-\infty}^{\infty} \xi^2 e^{-\xi^2}\,d\xi = \frac{\sqrt{\pi}}{2}$ (2)

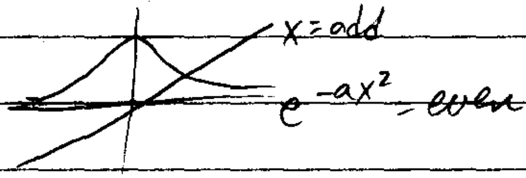
x^{even}, $a = 1$

$$1 = A_1^2\sqrt{\frac{\hbar}{m\omega}}\,\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi\hbar}{m\omega}}\,\frac{1}{2} = \sqrt{\frac{\pi\hbar}{4m\omega}}$$

$$A_1 = \left(\frac{4m\omega}{\pi\hbar}\right)^{1/4}$$

(d) $\displaystyle\int \psi_0\psi_1\,dx = A_0A_1\sqrt{\frac{m\omega}{\hbar}}\int x e^{-2m\omega x^2/2\hbar}\,dx = 0$ ✓

because



$$\int \psi_0\psi_2\,dx = A_0A_2\int e^{\frac{-m\omega x^2}{2\hbar}}(1 - 2b^2x^2)e^{\frac{-m\omega x^2}{2\hbar}}\,dx \quad \text{where } b = \sqrt{\frac{m\omega}{\hbar}}$$

$$= A_0A_2\left[\int e^{-\frac{m\omega x^2}{\hbar}}\,dx - 2b^2\int x^2 e^{-\frac{m\omega x^2}{\hbar}}\,dx\right]$$

Both these integrands are even, so we have to evaluate them

$$\int \psi_0\psi_2 = A_0A_2\left[\sqrt{\frac{\hbar}{m\omega}}\int e^{-\xi^2}\,d\xi - 2b^2\,\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}}\int \xi^2 e^{-\xi^2}\,d\xi\right]$$

(1) (2)

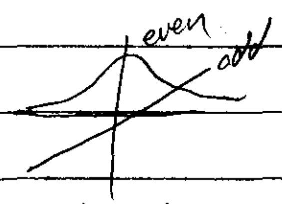
① $\int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi = \sqrt{\pi}$ and ② $\int_{-\infty}^{\infty} \xi^2 e^{-\xi^2}\,d\xi = \frac{\sqrt{\pi}}{2}$

So $$\int_{-\infty}^{\infty} \psi_0 \psi_2\,dx = A_0 A_2\left[\sqrt{\frac{\hbar}{m\omega}}\ ① - 2b^2 \frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}}\ ②\right]$$
$$= A_0 A_2\left[\sqrt{\frac{\hbar}{m\omega}}\sqrt{\pi} - 2\frac{m\omega}{\hbar}\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}}\frac{\sqrt{\pi}}{2}\right]$$
$$= A_0 A_2\left[\sqrt{\frac{\hbar\pi}{m\omega}} - \sqrt{\frac{\hbar\pi}{m\omega}}\right] = 0 \checkmark$$

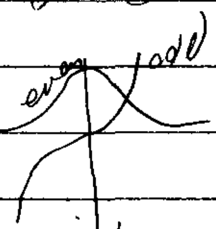
So ψ_0 and ψ_2 are also orthogonal.

$$\int_{-\infty}^{\infty} \psi_1 \psi_2\,dx = A_1 A_2 b \int_{-\infty}^{\infty} x(1-2b^2x^2) e^{-2ax^2}\,dx$$
$$= A_1 A_2 b\left[\int x e^{-2ax^2}\,dx - 2b^2 \int x^3 e^{-2ax^2}\,dx\right] = 0$$

because



integrates to 0



integrates to 0

So ψ_1 and ψ_2 are orthogonal

***Problem 2.14** Using the results of Problem 2.13,

(a) Compute $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, and $\langle p^2 \rangle$, for the states ψ_0 and ψ_1.

(b) Check the uncertainty principle for these states.

(c) Compute $\langle T \rangle$ and $\langle V \rangle$ for these states (no new integration allowed!). Is their sum what you would expect?

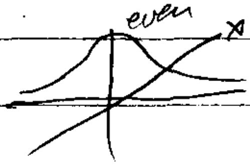
QHO

$\langle x \rangle = 0$ by symmetry

$A_0^2 = \sqrt{\frac{m\omega}{\pi\hbar}}$, $A_1^2 = 2\sqrt{\frac{m\omega}{\pi\hbar}}$

(a) $\int x\psi_0^2\,dx = A_0^2\int_{-\infty}^{\infty} x e^{-\frac{m\omega^2 x^2}{\hbar}}\,dx$ where $a = -\frac{m\omega}{2\hbar}$, $A_0 =$

even, x = odd



therefore $\langle x \rangle = 0$ ✓

ψ_0: $\langle p \rangle = m\frac{d\langle x \rangle}{dt} = 0$

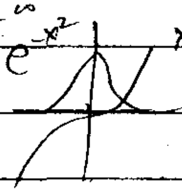
ψ_0: $\langle x^2 \rangle = A_0^2\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega^2 x^2}{\hbar}}\,dx = A_0^2\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}}\int \xi^2 e^{-\xi^2}\,d\xi$

$\langle x^2 \rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{2/4}\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}}\;\frac{\sqrt{\pi}}{2}$

$= \frac{1}{\sqrt{\pi}}\;\frac{\hbar}{m\omega}\;\frac{\sqrt{\pi}}{2} = \frac{\hbar}{2m\omega} = \langle x^2 \rangle$ ✓

ψ_1: $\int x\psi_1^2\,dx = A_1^2\frac{m\omega}{\hbar}\int_{-\infty}^{\infty} x\,x^2 e^{-}\,dx = 0$ because

even $= e^{-x^2}$, x^3 = odd



then $\langle p_1 \rangle = 0 = m\cdot\frac{d\langle x_1 \rangle}{dt}$ ✓

$\langle x^2 \rangle = A_1^2\frac{m\omega}{\hbar}\int x^4 e^{-\frac{m\omega^2 x^2}{\hbar}}\,dx$

$\int x^4 e^{-\xi^2}\,dx = \left(\frac{\hbar}{m\omega}\right)^2\sqrt{\frac{\hbar}{m\omega}}\int \xi^4 e^{-\xi^2}\,dx \to \frac{3\sqrt{\pi}}{2^2}$

$a = 2$:

$\langle x^2 \rangle = 2\sqrt{\frac{m\omega}{\pi\hbar}}\;\frac{m\omega}{\hbar}\left(\frac{\hbar}{m\omega}\right)^2\sqrt{\frac{\hbar}{m\omega}}\;3\sqrt{\pi}/4$

$\langle x^2 \rangle = \frac{3}{2}\hbar/m\omega$ ✓

ψ_0: $\langle p^2\rangle \neq 2mE$ because $V \neq 0$ here.

$p = -i\hbar \frac{\partial}{\partial x}$, $p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$

$\frac{\partial}{\partial x}\psi_0 = \frac{\partial}{\partial x} A_0 e^{-ax^2} = -2a A_0 e^{-ax^2}$ $\qquad a = {m\omega}/{2\hbar}$

$\frac{\partial^2}{\partial x^2}\psi_0 = 4a^2 A_0 e^{-ax^2}$

$\langle p_0^2\rangle = -\hbar^2 \int \psi_0 \frac{\partial^2\psi_0}{\partial x^2}\,dx = -\hbar^2 4a^2 A_0^2 \int e^{-\xi^2}\,dx$

$= -\left[\frac{(m\omega)^3}{\pi\hbar}\right]^{1/2} \int (mx^2\omega - \hbar)\, e^{-mx^2\omega/\hbar}\,dx = m\omega\hbar/2$

(See my Mathematica solutions)

ψ_1: $\frac{\partial\psi_1}{\partial x} = \frac{\partial}{\partial x} A_1 bxe^{-ax^2} = A_1 b\left(1 + x(-2ax)\right)e^{-ax^2}$
$= A_1 b(1-2ax^2)e^{-ax^2}$

$\frac{\partial^2\psi_1}{\partial x^2} = A_1 b e^{-ax^2}\left[(1-2ax^2)(-2ax) + (-4ax)\right]$

$[4a^2x^3 - 2ax - 4ax] = 4a^2x^3 - 6ax$

$\langle p_1^2\rangle = -\hbar^2 \int \psi_1 \frac{\partial^2\psi_1}{\partial x^2}\,dx = -\hbar^2 \int A_1 bxe^{-ax^2} A_1 b e^{-ax^2}(4a^2x^3 - 6ax)\,dx$

$= -\hbar^2 A_1^2 b^2 \int_{-\infty}^{\infty} e^{-2ax^2}(4a^2x^4 - 6ax^2)\,dx$

$= \frac{2\hbar}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{5/2} \int_{-\infty}^{\infty} e^{-mx^2\omega/\hbar}(mx^2\omega - 3\hbar)\,dx = 3\hbar^2\omega m/2\hbar$

(See my Mathematica solutions)

(b) Uncertainty principle:

ψ_0: $\langle x^2\rangle - \langle x\rangle^2 = (\Delta x)^2 = \frac{\hbar}{2m\omega}$

$\langle p^2\rangle - \langle p\rangle^2 = (\Delta p)^2 = \frac{\hbar\omega m}{2}$

$\Delta x\,\Delta p = \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\frac{\hbar\omega m}{2}} = \frac{\hbar}{2}$ ✓ $\geq \frac{\hbar}{2}$

ψ_1: $\langle x^2\rangle - \langle x\rangle^2 = (\Delta x)^2 = 3\hbar/2m\omega$

$\langle p^2\rangle - \langle p\rangle^2 = (\Delta p)^2 = \frac{3}{2}\hbar\omega m$

$\Delta x\,\Delta p = \sqrt{\frac{3\hbar}{2m\omega}}\sqrt{\frac{3\hbar\omega m}{2}} = \frac{3\hbar}{2}$ ✓ $\geq \frac{\hbar}{2}$

(c) $V = \frac{1}{2}kx^2$ where $F = ma = -kx = -m\omega^2 x \rightarrow k = m\omega^2$

$V = \frac{1}{2}m\omega^2 x^2$

$\langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle$ and $\langle T\rangle = \langle p^2\rangle/2m$

ψ_0: $\langle x^2\rangle = \frac{\hbar}{2m\omega}$ $\qquad \langle p^2\rangle = \frac{\hbar\omega m}{2}$

so $\langle V\rangle = \frac{m\omega^2}{2}\frac{\hbar}{2m\omega} = \frac{\hbar\omega}{4}$

EQUIPARTITION

$\langle T\rangle + \langle V\rangle = \frac{\hbar\omega}{2} = \langle E_0\rangle$ ✓

$\langle T\rangle = \frac{\hbar\omega m}{2}\frac{1}{2m} = \frac{\hbar\omega}{4}$

(c) for ψ_1, $\langle x^2 \rangle = \frac{3\hbar}{2m\omega}$ $\langle p^2 \rangle = \frac{3\hbar\omega m}{2}$

So $\langle V \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{m\omega^2}{2} \frac{3\hbar}{2m\omega} = \frac{3}{4}\hbar\omega$

and $\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{3\hbar m\omega}{4m} = \frac{3}{4}\hbar\omega$ EQUIPARTITION AGAIN

Are the sums what we expect? Should get

$\langle E \rangle = \langle T \rangle + \langle V \rangle$ where $\langle E \rangle = (n + \frac{1}{2})\hbar\omega$

ψ_0: $\langle E_0 \rangle = \frac{\hbar\omega}{2}$, $\langle T \rangle + \langle V \rangle = \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$ ✓

ψ_1: $\langle E_1 \rangle = \frac{3\hbar\omega}{2}$, $\langle T \rangle + \langle V \rangle = \frac{3}{4}\hbar\omega + \frac{3}{4}\hbar\omega = \frac{3}{2}\hbar\omega$ ✓

Quantum harmonic oscillator - EJZ March 2007

Calculate first two p^2 expectation values

In[1]:=
```
Clear[A0, A1];
A0 = Sqrt[Sqrt[m w/(π h)]];
a = m w/(2 h);
A1 = Sqrt[Sqrt[4 m w/(π h)]]
```

In[5]:=
```
Clear[Y0, Y1];
Y0[x_] := A0 Exp[-a x^2];
Y1[x_] := A1 * Sqrt[2 a] * x * Exp[-a x^2]
```

In[8]:=
```
Y0[x]
```

Out[8]= $\frac{e^{-\frac{m w x^2}{2 h}} \left(\frac{m w}{h}\right)^{1/4}}{\pi^{1/4}}$

$$< p_0{}^2 > = -hbar^2 \frac{d^2}{x^2}$$

In[9]:=
```
Clear [dY0, ddY0, dY1, ddY1];
dY0[x_] := Simplify[D[Y0[x], x]];
ddY0[x_] := Simplify[D[dY0[x], x]];
dY1[x_] := Simplify[D[Y1[x], x]];
ddY1[x_] := Simplify[D[dY1[x], x]];
```

In[14]:=
```
dY0[x]
```

Out[14]= $-\frac{e^{-\frac{m w x^2}{2 h}} \left(\frac{m w}{h}\right)^{5/4} x}{\pi^{1/4}}$

```
ddY1[x]
```

$\frac{\sqrt{2}\, e^{-\frac{m w x^2}{2 h}} \left(\frac{m w}{h}\right)^{7/4} x \left(-3 h + m w x^2\right)}{h\, \pi^{1/4}}$

In[15]:=
```
Clear[argp0, argp1];
argp0[x_] := Simplify[-h^2 Y0[x] ddY0[x]];
argp1[x_] := Simplify[-h^2 Y1[x] ddY1[x]];
```

In[18]:=
```
argp0[x]
```

Out[18]= $\frac{e^{-\frac{m w x^2}{h}} h \left(\frac{m w}{h}\right)^{3/2} \left(h - m w x^2\right)}{\sqrt{\pi}}$

```
argp1[x]
```

$$-\frac{2\, e^{-\frac{m w x^2}{h}}\, h \left(\frac{m w}{h}\right)^{5/2} x^2 \left(-3 h + m w x^2\right)}{\sqrt{\pi}}$$

Integrate each of these arguments over all space to find $< p^2 >$

■ Break p0 integral into parts to solve: $< p_0{}^2 > = \frac{h m w}{2}$

■ construct p0 argument

In[19]:= `argp0[x]`

Out[19]= $\frac{e^{-\frac{m w x^2}{h}}\, h \left(\frac{m w}{h}\right)^{3/2} \left(h - m w x^2\right)}{\sqrt{\pi}}$

```
Clear[c, i1, i2, int];
```

$$c = \frac{h \left(\frac{m w}{h}\right)^{3/2}}{\sqrt{\pi}};$$

$$i1[x\_] := e^{-\frac{m w x^2}{h}}\ (h);\quad i2[x\_] := e^{-\frac{m w x^2}{h}}\ (-m w x^2);$$

```
int[x_] := c (i1[x] + i2[x])
```

```
int[x]
```

$$\frac{h \left(\frac{m w}{h}\right)^{3/2} \left(e^{-\frac{m w x^2}{h}}\, h - e^{-\frac{m w x^2}{h}}\, m w x^2\right)}{\sqrt{\pi}}$$

In[25]:= `Simplify[int[x] - argp0[x]]`

Out[25]= 0

■ integrate first p0 argument

In[26]:= `Clear[s1];`
`s1 = Integrate[i1[x], {x, -∞, ∞}]`

Out[27]= $h\, \mathrm{If}\left[\mathrm{Re}\left[\frac{m w}{h}\right] > 0,\ \frac{\sqrt{\pi}}{\sqrt{\frac{m w}{h}}},\ \int_{-\infty}^{\infty} e^{-\frac{m w x^2}{h}}\, dx\right]$

In[28]:= $s1 = \mathrm{Simplify}\left[h\, \frac{\sqrt{\pi}}{\sqrt{\frac{m w}{h}}}\right]$

■ integrate second p0 argument

```
In[29]:= Clear[s2];
         Integrate[i2[x], {x, -∞, ∞}]
```

Out[30]= $-\mathrm{m\,w\,If}\left[\mathrm{Re}\left[\frac{\mathrm{m\,w}}{\mathrm{h}}\right]>0,\ \frac{\sqrt{\pi}}{2\left(\frac{\mathrm{m\,w}}{\mathrm{h}}\right)^{3/2}},\ \int_{-\infty}^{\infty}e^{-\frac{\mathrm{m\,w}\,x^2}{\mathrm{h}}}x^2\,dx\right]$

In[31]:= $\mathbf{s2} = \mathbf{Simplify}\left[\frac{-\mathrm{m\,w}\sqrt{\pi}}{2\left(\frac{\mathrm{m\,w}}{\mathrm{h}}\right)^{3/2}}\right]$

■ Combine into $< p_0{}^2 >$

```
In[34]:= Clear[sum];
         sum = Simplify[s1 + s2]
```

Out[35]= $\frac{\mathrm{h}\sqrt{\pi}}{2\sqrt{\frac{\mathrm{m\,w}}{\mathrm{h}}}}$

```
In[33]:= c
```

```
In[36]:= Simplify[c * sum]
```

Out[36]= $\frac{\mathrm{h\,m\,w}}{2}$

■ Break p1 integral into parts to solve: $< p_1{}^2 > = \frac{3\,\mathrm{h\,m\,w}}{2}$

```
argp1[x]
```

$-\frac{2\,e^{-\frac{\mathrm{m\,w}\,x^2}{\mathrm{h}}}\,\mathrm{h}\left(\frac{\mathrm{m\,w}}{\mathrm{h}}\right)^{5/2}x^2\left(-3\,\mathrm{h}+\mathrm{m\,w}\,x^2\right)}{\sqrt{\pi}}$

■ construct p1 argument

```
Clear[c, i1, i2, int];
```

$\mathbf{c} = -\frac{2\,\mathrm{h}\left(\frac{\mathrm{m\,w}}{\mathrm{h}}\right)^{5/2}}{\sqrt{\pi}};$

$\mathbf{i1}[\mathbf{x\_}] := e^{-\frac{\mathrm{m\,w}\,x^2}{\mathrm{h}}}\,x^2\,(-3\,\mathrm{h});\quad \mathbf{i2}[\mathbf{x\_}] := e^{-\frac{\mathrm{m\,w}\,x^2}{\mathrm{h}}}\,x^2\,(\mathrm{m\,w}\,x^2);$

```
int[x_] := c (i1[x] + i2[x]);
int[x]
```

$-\frac{2\,\mathrm{h}\left(\frac{\mathrm{m\,w}}{\mathrm{h}}\right)^{5/2}\left(-3\,e^{-\frac{\mathrm{m\,w}\,x^2}{\mathrm{h}}}\,\mathrm{h}\,x^2+e^{-\frac{\mathrm{m\,w}\,x^2}{\mathrm{h}}}\,\mathrm{m\,w}\,x^4\right)}{\sqrt{\pi}}$

```
Simplify[int[x] - argp1[x]]
```

0

integrate first p1 argument

```
Clear[s1];
s1 = Integrate[i1[x], {x, -∞, ∞}]
```

$$-3\,h\,\mathrm{If}\left[\mathrm{Re}\left[\frac{m\,w}{h}\right] > 0,\ \frac{\sqrt{\pi}}{2\left(\frac{m\,w}{h}\right)^{3/2}},\ \int_{-\infty}^{\infty} e^{-\frac{m\,w\,x^2}{h}}\,x^2\,dx\right]$$

$$\mathbf{s1 = Simplify}\left[-3\,h\,\frac{\sqrt{\pi}}{2\left(\frac{m\,w}{h}\right)^{3/2}}\right]$$

integrate second p0 argument

```
Clear[s2];
Integrate[i2[x], {x, -∞, ∞}]
```

$$m\,w\,\mathrm{If}\left[\mathrm{Re}\left[\frac{m\,w}{h}\right] > 0,\ \frac{3\sqrt{\pi}}{4\left(\frac{m\,w}{h}\right)^{5/2}},\ \int_{-\infty}^{\infty} e^{-\frac{m\,w\,x^2}{h}}\,x^4\,dx\right]$$

$$\mathbf{s2 = Simplify}\left[m\,w\,\frac{3\sqrt{\pi}}{4\left(\frac{m\,w}{h}\right)^{5/2}}\right]$$

Combine into $< p_1{}^2 >$

```
Clear[sum];
sum = Simplify[s1 + s2]
```

$$-\frac{3\,h\,\sqrt{\pi}}{4\left(\frac{m\,w}{h}\right)^{3/2}}$$

```
c
```

```
Simplify[c * sum]
```

$$\frac{3\,h\,m\,w}{2}$$