Vector Calculus HW #4 due Tues 20 Feb 2007 - E/Z

Ch 1.4 #37, 38, 41, 42

Problem 1.37 Express the unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.64). Check your answers several ways ($\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \stackrel{?}{=} 1$, $\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} \stackrel{?}{=} 0$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \stackrel{?}{=} \hat{\boldsymbol{\phi}}, \ldots$). Also work out the inverse formulas, giving $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ (and θ, ϕ). @ Done in class

$$\left.\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \end{aligned}\right\} \qquad (1.64)$$

There are many ways to do this one—probably the most illuminating way is to work it out by trigonometry from Fig. 1.36. The most systematic approach is to study the expression:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}.$$

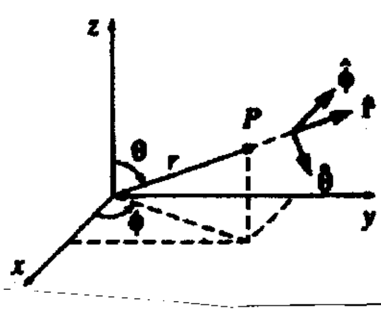
If I only vary r slightly, then $d\mathbf{r} = \frac{\partial}{\partial r}(\mathbf{r})dr$ is a short vector pointing in the direction of increase in r. To make it a unit vector, I must divide by its length. Thus:

$$\hat{\mathbf{r}} = \frac{\frac{\partial \mathbf{r}}{\partial r}}{\left|\frac{\partial \mathbf{r}}{\partial r}\right|};\ \hat{\boldsymbol{\theta}} = \frac{\frac{\partial \mathbf{r}}{\partial \theta}}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|};\ \hat{\boldsymbol{\phi}} = \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left|\frac{\partial \mathbf{r}}{\partial \phi}\right|}.$$

$\frac{\partial \mathbf{r}}{\partial r} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$; $\left|\frac{\partial \mathbf{r}}{\partial r}\right|^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1$.
$\frac{\partial \mathbf{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{\mathbf{x}} + r\cos\theta\sin\phi\,\hat{\mathbf{y}} - r\sin\theta\,\hat{\mathbf{z}}$; $\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|^2 = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2$.
$\frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{\mathbf{x}} + r\sin\theta\cos\phi\,\hat{\mathbf{y}}$; $\left|\frac{\partial \mathbf{r}}{\partial \phi}\right|^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta$.

$$\Rightarrow \boxed{\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}. \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}. \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}. \end{aligned}}$$

Check: $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$, ✓
$\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = -\cos\theta\sin\phi\cos\phi + \cos\theta\sin\phi\cos\phi = 0$, ✓ etc.



$\sin\theta\,\hat{\mathbf{r}} = \sin^2\theta\cos\phi\,\hat{\mathbf{x}} + \sin^2\theta\sin\phi\,\hat{\mathbf{y}} + \sin\theta\cos\theta\,\hat{\mathbf{z}}$.
$\cos\theta\,\hat{\boldsymbol{\theta}} = \cos^2\theta\cos\phi\,\hat{\mathbf{x}} + \cos^2\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\cos\theta\,\hat{\mathbf{z}}$.

Add these:

(1) $\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} = +\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$;
(2) $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$.

Multiply (1) by $\cos\phi$, (2) by $\sin\phi$, and subtract:

$$\boxed{\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}.}$$

Multiply (1) by $\sin\phi$, (2) by $\cos\phi$, and add:

$$\boxed{\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}.}$$

$\cos\theta\,\hat{\mathbf{r}} = \sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} + \cos^2\theta\,\hat{\mathbf{z}}$.
$\sin\theta\,\hat{\boldsymbol{\theta}} = \sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin^2\theta\,\hat{\mathbf{z}}$.

Subtract these:

$$\boxed{\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.}$$

Problem 1.38 $\int_{volume} (\nabla \cdot \bar{v})\, d\tau = \oint_{area} \bar{v} \cdot d\bar{a}$

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2 \hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

b) Do the same for $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob. 1.16.)

Spherical coordinates -

$$\bar{\nabla} \cdot \bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

(a) $\bar{\nabla} \cdot \bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) = \frac{1}{r^2}(4r^3) = 4r$

$\int (\bar{\nabla} \cdot \bar{v})\, d\tau = \int 4r\, d\tau$ where $d\tau = r^2\, dr \sin\theta\, d\theta\, d\phi$

$\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = -2\pi\cos\theta \Big|_0^\pi = -2\pi(-1-1) = 4\pi$

$= \int_0^R 4r\; 4\pi r^2\, dr$

$= 16\pi \int_0^R r^3\, dr = 16\pi \frac{r^4}{4}\Big|_0^R$

$\int (\bar{\nabla} \cdot \bar{v})\, d\tau = 4\pi R^4$.

Now check $\oint \bar{v} \cdot d\bar{a}$ where $da = r^2 \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi r^2$

$\oint \bar{v} \cdot d\bar{a} = r^2\, 4\pi r^2 \Big|_{r=R} = 4\pi R^4$ ✓

(b) $\bar{\nabla} \cdot \bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\left(\frac{\partial}{\partial r} 1\right) = 0$, so $\int (\bar{\nabla} \cdot \bar{v})\, d\tau = 0$

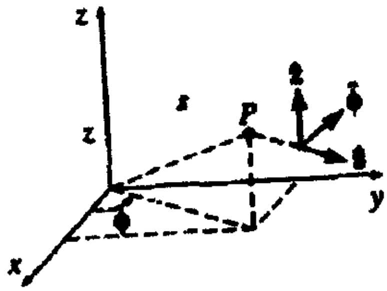
$\oint \bar{v} \cdot d\bar{a} = \frac{1}{r^2}\, 4\pi r^2 = 4\pi$? These don't agree?!

$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 0$ for $r \neq 0$
$= \infty$ at $r = 0$

$\int \left(\bar{\nabla} \cdot \frac{\hat{r}}{r^2}\right) d\tau = 4\pi$ is correct

Problem 1.41 Express the cylindrical unit vectors $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.75). "Invert" your formulas to get $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ (and ϕ).

$$\left.\begin{aligned} \hat{\mathbf{s}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}. \end{aligned}\right\} \qquad (1.75)$$

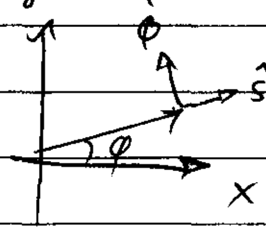


As I showed in class, from Fig 1.42,

$\hat{s} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$

$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

$\hat{z} = \hat{z}$



$\hat{s}\cos\varphi = \cos^2\varphi\,\hat{x} + \cos\varphi\sin\varphi\,\hat{y}$

$\hat{\varphi}\sin\varphi = -\sin^2\varphi\,\hat{x} + \cos\varphi\sin\varphi\,\hat{y}$

subtract these two equations:

$\hat{s}\cos\varphi + \hat{\varphi}\sin\varphi = (\cos^2\varphi + \sin^2\varphi)\,\hat{x} + (\cos\varphi\sin\varphi - \cos\varphi\sin\varphi)\,\hat{y}$

$= \quad 1 \quad \hat{x} \quad + \quad 0 \quad \hat{y}$

$\underline{\hat{s}\cos\varphi + \hat{\varphi}\sin\varphi = \hat{x}}$

$\hat{S}\sin\varphi = \cos\varphi\sin\varphi\,\hat{x} + \sin^2\varphi\,\hat{y}$

$\hat{\varphi}\cos\varphi = -\cos\varphi\sin\varphi\,\hat{x} + \cos^2\varphi\,\hat{y}$

add these two equations:

$\hat{s}\sin\varphi + \hat{\varphi}\cos\varphi = \quad 0\,\hat{x} \quad + \quad 1\,\hat{y} \rightarrow \underline{\hat{y} = \hat{s}\sin\varphi + \hat{\varphi}\cos\varphi}$

And we already have $\hat{z} = \hat{z}$.

Problem 1.42

(a) Find the divergence of the function

$$\mathbf{v} = s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}.$$

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\frac{\partial V_z}{\partial z} = 3$$

$$\frac{\partial v_\varphi}{\partial \varphi} = s\left[\sin\varphi(-\sin\varphi) + \cos\varphi(\cos\varphi)\right] = s\left[\cos^2\varphi - \sin^2\varphi\right]$$

$$\frac{\partial}{\partial s}(sV_s) = \frac{\partial}{\partial s}\, s^2(2+\sin^2\varphi) = 2s(2+\sin^2\varphi)$$

$$\nabla\cdot V = 2(2+\sin^2\varphi) + [\cos^2\varphi - \sin^2\varphi] + 3$$

$$= 4 + 2\sin^2\varphi - \sin^2\varphi + \cos^2\varphi + 3$$

$$= 4 + \sin^2\varphi + \cos^2\varphi + 3$$

$$= 4 + 1 + 3 = 8$$

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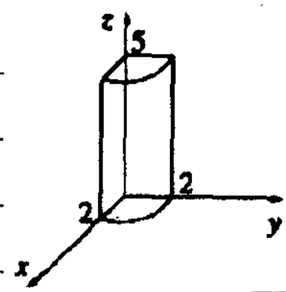
(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.



$$\int_{\mathcal{V}} (\nabla\cdot\mathbf{v})\,d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a}$$

$$\int d\tau = \int_0^2 s\,ds \int_0^{\pi/2} d\varphi \int_0^5 dz$$

$$\int(\nabla\cdot V)\,d\tau = \int_0^2 8\,s\,ds \int_0^{\pi/2} d\varphi \int_0^5 dz$$

$$= 8\cdot 2\cdot \frac{\pi}{2}\cdot 5 = 8\pi\cdot 5 = 40\pi$$

$$d\bar{a}_{top} = s\,ds\,d\varphi\,\hat{z}, \quad z = 5$$

$$\bar{V}\cdot d\bar{a} = 3z\,s\,ds\,d\varphi = 15\,s\,ds\,d\varphi$$

$$\int \bar{V}\cdot d\bar{a} = 15\int_0^2 s\,ds \int_0^{\pi/2} d\varphi = 15\cdot 2\cdot\frac{\pi}{2} = \underline{15\pi}$$

TOP

$da_{bottom} = -s\, ds\, d\varphi\, \hat{z}$, $z=0$

$\bar{V}\cdot d\bar{a} = -3z s\, ds\, d\varphi = 0 \rightarrow \int V\cdot da = 0$ / bottom

$da_{front} = s\, d\varphi\, dz\, \vec{s}$, $s=2$

$\bar{V}\cdot d\bar{a} = V_s \cdot da = s(2+\sin^2\varphi)\, s\, d\varphi\, dz = 4(2+\sin^2\varphi)\, d\varphi\, dz$

$\int \bar{V}\cdot d\bar{a} = 4\int_0^{\pi/2}(2+\sin^2\varphi)\, d\varphi \int_0^5 dz = 20\left(\pi+\frac{\pi}{4}\right) = 20\cdot\frac{5}{4}\pi$

$= 25\pi$/ front

$da_{back} = ds\, dz\, \vec{\varphi}$, $\varphi = \frac{\pi}{2}$

$\bar{V}\cdot d\bar{a} = V_\varphi \cdot da = s\sin\varphi\cos\varphi\, ds\, dz = 0 \rightarrow \int V\cdot da = 0$/ back

$da_{left\ side} = -ds\, dz\, \vec{\varphi}$, $\varphi = 0$

$\bar{V}\cdot d\bar{a} = \bar{V_\varphi}\cdot d\bar{a} = -s\sin\varphi\cos\varphi\, ds\, dz = 0 \rightarrow \int V\cdot da = 0$/ side

$\int \bar{V}\cdot d\bar{a} = 15\pi + 25\pi = 40\pi = \int(\nabla\cdot V)\, d\tau$ ✓

Satisfies divergence Thm

(c) ...

1.12

(c) Find the curl of v. $v = s(2+\sin^2\varphi)\,\hat{s} + s\sin\varphi\cos\varphi\,\hat{\varphi} + 3z\,\hat{z}$

$$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\frac{\partial v_\varphi}{\partial z} = \frac{\partial}{\partial z}\, s\sin\varphi\cos\varphi = 0$$

$$\frac{\partial v_z}{\partial \varphi} = \frac{\partial}{\partial \varphi}\, 3z = 0$$

$$[\bar{\nabla}\times\bar{v}]_s = 0$$

$$\frac{\partial v_z}{\partial s} = \frac{\partial}{\partial s}\, 3z = 0$$

$$\frac{\partial v_s}{\partial z} = \frac{\partial}{\partial z}\, s(2+\sin^2\varphi) = 0$$

$$[\bar{\nabla}\times\bar{v}]_\varphi = 0$$

$$\frac{\partial v_s}{\partial \varphi} = \frac{\partial}{\partial \varphi}\, s(2+\sin^2\varphi) = s(2\sin\varphi\cos\varphi)$$

$$\frac{\partial}{\partial s}(s v_\varphi) = \frac{\partial}{\partial s}(s^2\sin\varphi\cos\varphi) = 2s\sin\varphi\cos\varphi$$

$$[\vec{\nabla}\times\vec{v}]_z = \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\varphi) - \frac{\partial v_s}{\partial \varphi}\right] = 0$$

$\bar{\nabla}\times\bar{v} = 0$: $\bar{v}$ could represent a conservative field.