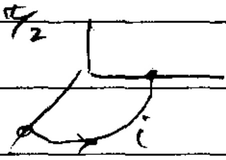
Problem 1.56 Compute the line integral of $\quad d\vec{l} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$

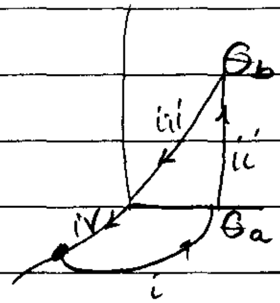
$$\mathbf{v} = (r\cos^2\theta)\,\hat{\mathbf{r}} - (r\cos\theta\sin\theta)\,\hat{\boldsymbol{\theta}} + 3r\,\hat{\boldsymbol{\phi}}$$

around the path shown in Fig. 1.50 (the points are labeled by their Cartesian coordinates). Do it either in cylindrical or in spherical coordinates. Check your answer, using Stokes' theorem. [*Answer:* $3\pi/2$]

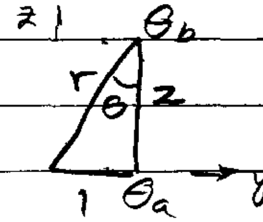
$r = 1 \quad \theta = \frac{\pi}{2}, \quad \phi\Big|_0^{\pi/2}$ $\quad d l\,\hat{\phi}$



(i) $dr = 0, \; d\theta = 0,$



$$\int \vec{v}\cdot d\vec{l} = \int 3r\cdot r\sin\theta\, d\phi = 3r^2\Big|_{r=1} \sin\tfrac{\pi}{2}\; \phi\Big|_0^{\pi/2}$$
$$= 3\cdot 1\cdot \pi/2 = 3\pi/2$$



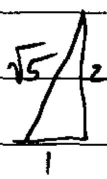
$r = \frac{1}{\sin\theta}$

$\frac{1}{r} = \sin\theta$

(ii) $d\phi = 0, \quad \theta\Big|_{\pi/2}^{\theta_b}, \quad r\Big|_1^{\frac{1}{\sin\theta}}$ $\quad dr = \frac{-\cos\theta}{\sin^2\theta}\,d\theta$

$\phi = \frac{\pi}{2}$

$\frac{\sin\theta}{\cos\theta} = \frac{1}{2} = \tan\theta_b$



$\theta_b = 26.56°$

(iii) $r\Big|_{\sqrt{5}}^{0}, \quad d\phi = 0, \quad d\theta = 0$

$\phi = \pi/2 \quad \theta = \theta_b$

$\sin\theta_b = 0.447$

$\sin^2\theta_b = 0.200$

$\cos\theta_b = 0.894$

(iv) $r\Big|_0^1, \quad d\theta = d\phi = 0$ $\quad \cos^2\theta_b = 0.800$

$\theta = \frac{\pi}{2}, \quad \phi = 0$

(ii) $\int \vec{v}\cdot d\vec{l} = \int v_r\, dl_r + \int v_\theta\, dl_\theta = \int r\cos^2\theta\, dr - \int (r\cos\theta\sin\theta)\, r\, d\theta$

$$= \int \frac{\cos^2\theta}{\sin\theta}\left(\frac{-\cos\theta}{\sin^2\theta}\right) d\theta - \int \frac{\cos\theta\sin\theta}{\sin^2\theta}\, d\theta$$

$$\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta}{\sin^2\theta} + 1\cdot\frac{\sin^2\theta}{\sin^2\theta}\right) = \frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)$$

$$\int \vec{v}\cdot d\vec{l} = \int \frac{\cos\theta}{\sin^3\theta}\, d\theta = \frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\theta_b} = \frac{1}{2}\left(\frac{1}{0.20} - \frac{1}{1} = 5 - 1 = 4\right) = \frac{4}{2} = 2$$

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(iii) $\int \vec{v}\cdot d\vec{l} = \int v_r\, dr = \int r\cos^2\theta\, dr = \frac{r^2}{2}\cos^2\theta_6 = \frac{0.8}{2} r^2 \Big|_{\sqrt{5}}^{0}$

$= 0.4\,(0-5)$

$= -2$

(iv) $\int \vec{v}\cdot d\vec{l} = \int v_r\, dr = \int r\cos^2\theta\, dr = \frac{r^2}{2}\cos^2\frac{\pi}{2} = 0$

$\oint \vec{v}\cdot d\vec{l} = \underset{(i)}{\frac{3\pi}{2}} + \underset{(ii)}{2} - \underset{(iii)}{2} + \underset{(iv)}{0} = \frac{3\pi}{2}$

To check with Stokes' theorem, I need $\nabla\times\vec{v}$
In spherical coordinates:

$(\nabla\times v)_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta\, v_\phi - \frac{\partial v_\theta}{\partial\phi}\right]$

$\frac{\partial v_\theta}{\partial\phi} = \frac{\partial}{\partial\phi}(-r\cos\theta\sin\theta) = 0$

$\frac{\partial}{\partial\theta}\sin\theta\, v_\phi = \frac{\partial}{\partial\theta}\sin\theta\, 3r = 3r\cos\theta$

$(\nabla\times v)_r = \frac{3r\cos\theta}{r\sin\theta} = 3\frac{\cos\theta}{\sin\theta}$

$(\nabla\times v)_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]$

$\frac{\partial}{\partial r}(r v_\phi) = \frac{\partial}{\partial r}(r\cdot 3r) = \frac{\partial}{\partial r}3r^2 = 6r$

$\frac{\partial v_r}{\partial\phi} = \frac{\partial}{\partial\phi} r\cos^2\theta = 0 \rightarrow (\nabla\times v)_\theta = \frac{1}{r}(-6r$

$(\nabla\times v)_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right] \quad \frac{\partial v_r}{\partial\theta} = \frac{\partial}{\partial\theta} r\cos^2\theta = -2r\cos\theta\sin\theta$

$\frac{\partial}{\partial r}(r v_\theta) = \frac{\partial}{\partial r}(-r^2\cos\theta\sin\theta) = -2r\cos\theta\sin\theta \rightarrow (\nabla\times v)_\phi =$

1.56 continued: Stokes' theorem: $\oint \bar{v} \cdot d\bar{l} = \int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a}$

$(\bar{\nabla} \times \bar{v}) = 3\frac{\cos\theta}{\sin\theta}\,\hat{r} - 6\hat{\theta} + 0\hat{\phi}$

Back face: $d\bar{a} = -r\,dr\,d\theta\ \hat{\phi}$, $(\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = 0$

Bottom: $d\bar{a} = -r\sin\theta\,dr\,d\phi\,\hat{\theta}$, $(\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = +6r\sin\theta\,dr\,d\phi$

$\theta = \pi/2$

$$\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = \int_{r=0}^{1} 6r\,dr \int_{0}^{\pi/2} d\phi$$

$$= \frac{6r^2}{2}\Bigg|_0^1 \phi\Bigg|_0^{\pi/2}$$

$$= 3 \cdot 1 \cdot \pi/2 = 3\pi/2 = \int \vec{v} \cdot d\vec{l}$$

Stokes' theorem checks.