Vector Calculus HW #6 due Tues 6 Mar 2007 E7Z

Ch 1.6 # 49, 52, 53, 56

Problem 1.49

(a) Let $\mathbf{F}_1 = x^2\,\hat{\mathbf{z}}$ and $\mathbf{F}_2 = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$. Calculate the divergence and curl of $\mathbf{F}_1$ and $\mathbf{F}_2$. Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.

$$\bar{\nabla}\cdot\bar{F}_1 = \frac{\partial}{\partial z}x^2 = 0 \therefore \bar{F}_1 = \bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$F_x = 0 \therefore \frac{\partial A_z}{\partial y} = \frac{\partial A_y}{\partial z}$

$F_y = 0 \therefore \frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial z}$

$F_z = x^2 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$; Let $A_x = 0$, then $\int dA_y = \int x^2\,dx$

Can let $A_z = 0$ also $\qquad A_y = \frac{x^3}{3} \rightarrow \vec{A}_1 = \frac{x^3}{3}\hat{y}$

$$\nabla\times\bar{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = -\hat{y}\frac{\partial}{\partial x}x^2 = -2x\,\hat{y} \neq 0 \therefore F_1 \neq \nabla U$$

$$\bar{\nabla}\cdot\bar{F}_2 = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \neq 0 \therefore \bar{F}_2 \neq \bar{\nabla}\times\bar{A}$$

$$\nabla\times\bar{F}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \therefore \bar{F}_2 = \bar{\nabla}U = \hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z} = \hat{x}\,x + \hat{y}\,y + \hat{z}\,z$$

$x = \frac{\partial U}{\partial x} \rightarrow U = \int x\,dx = \frac{x^2}{2}$, similarly $U = \frac{y^2}{2}$, $U = \frac{z^2}{2}$

$U_2 = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + \text{constant}$

1.49

(b) Show that $\mathbf{F}_3 = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

$$\bar{\nabla}\times\bar{F}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right) - \hat{y}\left(\frac{\partial}{\partial x}xy - \frac{\partial}{\partial z}yz\right) + \hat{z}\left(\frac{\partial xz}{\partial x} - \frac{\partial yz}{\partial y}\right)$$

$$= \hat{x}(x-x) - \hat{y}(y-y) + \hat{z}(z-z) = 0 \checkmark$$

Therefore $F_3 = \nabla u = \frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y} + \frac{\partial u}{\partial z}\hat{z}$

$\frac{\partial u}{\partial x} = yz \qquad \frac{\partial u}{\partial y} = xz \qquad \frac{\partial u}{\partial z} = xy$

$u = xyz \qquad u = xyz \qquad u = xyz \checkmark$

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$
$$= \frac{\partial}{\partial x}yz + \frac{\partial}{\partial y}xz + \frac{\partial}{\partial z}xy$$
$$= 0 + 0 + 0$$

Therefore $\bar{F}_3 = \bar{\nabla}\times\bar{A}$

$F_x = (\bar{\nabla}\times\bar{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz$

$F_y = (\bar{\nabla}\times\bar{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz$

$F_z = (\bar{\nabla}\times A)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy$

This is so symmetric, I may need to keep all terms. Try getting half from each.

x: $\frac{\partial A_z}{\partial y} = \frac{1}{2}yz \rightarrow A_z = \int \frac{1}{2}yz\,dy = \frac{y^2}{4}z$

$-\frac{\partial A_y}{\partial z} = \frac{1}{2}yz \rightarrow A_y = -\int \frac{1}{2}yz\,dz = -\frac{yz^2}{4}$

y: $\frac{\partial A_x}{\partial z} = \frac{1}{2}xz \rightarrow A_x = \frac{xz^2}{4}, \quad -\frac{\partial A_z}{\partial x} = \frac{xz}{2} \rightarrow A_z = -\frac{x^2 z}{4}$

z: $\frac{\partial A_y}{\partial x} = \frac{xy}{2} \rightarrow A_y = \frac{x^2 y}{4}, \quad \frac{\partial A_x}{\partial y} = -\frac{xy}{2} \rightarrow A_x = -\frac{xy^2}{4}$

Put these together: $A_x = \frac{xz^2 - xy^2}{4}, \; A_y = \frac{x^2 y - yz^2}{4}, \; A_z = \frac{zy^2 - zx^2}{4}$

Then check

Problem 1.52

(a) Which of the vectors in Problem 1.15 can be expressed as the gradient of a scalar? Find a scalar function that does the job.

(b) Which can be expressed as the curl of a vector? Find such a vector.

Problem 1.15 Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$. $\bar{\nabla}\cdot\bar{V}_a = 0$ from problem (1.15) $\rightarrow$ $\bar{V}_a = \bar{\nabla}\times\bar{A}$

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$. $\bar{\nabla}\times\bar{V}_c = 0$ from problem (1.18)

$\rightarrow \bar{V}_c = \bar{\nabla} U$

(a) $(\bar{\nabla}\times\bar{A})_x = V_{ax} = x^2 = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$

$(\nabla\times A)_y = V_{ay} = 3xz^2 = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$

$(\nabla\times A)_z = V_{az} = -2xz = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$

y: If we let $A_x = 0$, then $-A_z = \int 3xz^2\,dx = \frac{3}{2}x^2z^2$

x: $\frac{\partial A_z}{\partial y} = 0$ so $A_y = -\int x^2\,dz = -x^2 z$

z: Check these: $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial}{\partial x}(-x^2 z) - \frac{\partial}{\partial y}(0) = -2xz$ ✓

So $\vec{A} = (-x^2 z)\,\hat{y} - \left(\frac{3}{2}x^2z^2\right)\hat{z}$

(c) $\bar{V}_c = \bar{\nabla} U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}$

$\frac{\partial U}{\partial x} = y^2$, $(2xy + z^2) = \frac{\partial U}{\partial y}$, $\frac{\partial U}{\partial z} = 2yz$

$U = \int 2yz\,dz = 2y\frac{z^2}{2} = yz^2$

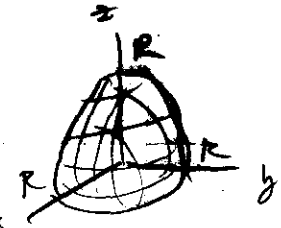
$U = \int y^2\,dx = xy^2$

If $U = xy^2 + yz^2$, check $\frac{\partial U}{\partial y} = 2xy + z^2$ ✓

Problem 1.53 Check the divergence theorem for the function

$$\mathbf{v} = r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}},$$

using as your volume one octant of the sphere of radius R (Fig. 1.48). Make sure you include the *entire* surface. [*Answer:* $\pi R^4/4$]



Divergence theorem: $\int(\bar\nabla\cdot\bar v)\,d\tau = \int \bar v\cdot d\bar a$

In spherical coordinates, the divergence is

$$\bar\nabla\cdot\bar v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

For this function

$$\bar\nabla\cdot\bar v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= \frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{r^2\cos\phi}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta - \frac{r^2\cos\theta}{r\sin\theta}\frac{\partial}{\partial\phi}\sin\phi$$

$$= \frac{\cos\theta}{r^2}(4r^3) + \underbrace{\frac{r\cos\phi}{\sin\theta}(\cos\theta) - \frac{r\cos\theta}{\sin\theta}(\cos\phi)}_{\text{these cancel}}$$

$$\bar\nabla\cdot\bar v = 4r\cos\theta$$

$$\int(\bar\nabla\cdot\bar v)\,d\tau = \int(4r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi = 4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

$$= 4\,\frac{R^4}{4}\,\frac{\pi}{2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

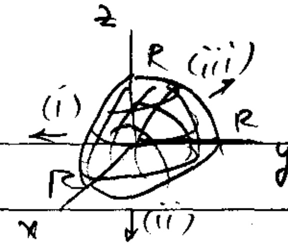
Dwight $\frac{450.11}{106}$ $\int\sin x\cos x\,dx = \frac{\sin^2 x}{2} = \frac{-\cos^2 x}{2} + \text{const} = \frac{-\cos 2x}{4} + \text{const}$

$$\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \tfrac12\sin^2\theta\Big|_0^{\pi/2} = \tfrac12(1^2-0^2) = \tfrac12$$

$$\int(\bar\nabla\cdot\bar v)\,d\tau = \frac{\pi R^4}{2}\left(\frac12\right) = \frac{\pi R^4}{4}$$

Now check $\int\bar v\cdot d\bar a$, which has four surfaces for $d\bar a$

1.53 continued $\bar{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$



(i) Left side = x - z plane, $y = 0$

$\phi = 0$, $d\bar{a} = -r\,dr\,d\theta\,\hat{\phi}$

$$\int \bar{v}\cdot d\bar{a} = \int \underbrace{(-r^2\cos\theta\sin\phi)}_{\phi=0}(-r\,dr\,d\theta) = 0$$

(ii) Bottom = x - y plane, $z=0$, $\theta = \frac{\pi}{2}$

$d\bar{a} = r\,dr\,d\phi\,\hat{\theta}$

$$\int \bar{v}\cdot d\bar{a} = \int (r^2\cos\phi)(r\,dr\,d\phi) = \int_0^R r^3\,dr \int_0^{\pi/2}\cos\phi\,d\phi$$

$$= \frac{r^4}{4}\Big|_0^R (+\sin\phi)\Big|_0^{\pi/2} = \frac{R^4}{4}\cdot(1-0) = \frac{R^4}{4}$$

(iii) Back side, y - z plane, $x = 0$, $\phi = \frac{\pi}{2}$

$d\bar{a} = r\,dr\,d\theta\,\hat{\phi}$

$$\int \bar{v}\cdot d\bar{a} = \int \underbrace{(-r^2\cos\theta\sin\phi)}_{\sin\frac{\pi}{2}=1}(r\,dr\,d\theta) = -\int_0^R r^3\,dr\ \sin\phi \int_0^{\pi/2}\cos\theta\,d\theta$$

$$= -\frac{R^4}{4}(1)(1) = -\frac{R^4}{4}$$

(iv) Front curved surface; $r = R$, $d\bar{a} = dl_\theta\, dl_\phi\,\hat{r}$

$d\bar{a} = (R\,d\theta)(R\sin\theta\,d\phi)\,\hat{r}$

$$\int \bar{v}\cdot d\bar{a} = \int (R^2\cos\theta)(R^2\sin\theta\,d\theta\,d\phi)$$

$$= R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi = \frac{\pi}{2}R^4\,\frac{1}{2}\sin^2\theta\Big|_0^{\pi/2}$$

$$= \frac{\pi R^4}{4}$$

$$\oint \bar{v}\cdot d\bar{a} = \frac{\pi R^4}{4} = \int (\bar{\nabla}\cdot\bar{v})\,d\tau \checkmark$$

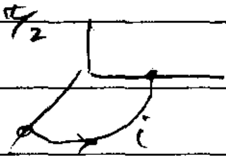
Divergence theorem is satisfied.

Problem 1.56 Compute the line integral of $d\vec{l} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$

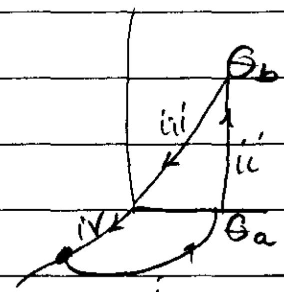
$$\mathbf{v} = (r\cos^2\theta)\,\hat{\mathbf{r}} - (r\cos\theta\sin\theta)\,\hat{\boldsymbol{\theta}} + 3r\,\hat{\boldsymbol{\phi}}$$

around the path shown in Fig. 1.50 (the points are labeled by their Cartesian coordinates). Do it either in cylindrical or in spherical coordinates. Check your answer, using Stokes' theorem. [*Answer:* $3\pi/2$]

$r=1 \quad \theta = \frac{\pi}{2}, \ \phi\Big|_0^{\pi/2}$ $\qquad dl\,\hat{\phi}$

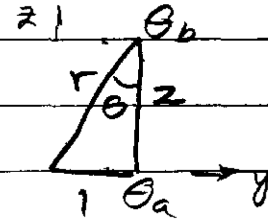


(i) $dr=0,\ d\theta=0,$



$\int \vec{v}\cdot d\vec{l} = \int 3r\cdot r\sin\theta\, d\phi = 3r^2\Big|_{r=1} \sin\frac{\pi}{2}\ \phi\Big|_0^{\pi/2}$

$= 3\cdot 1\cdot \pi/2 = 3\pi/2$



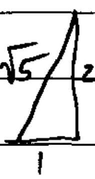
$r = \frac{1}{\sin\theta}$

$\frac{1}{r} = \sin\theta$

(ii) $d\phi = 0, \ \theta\Big|_{\pi/2}^{\theta_b}, \ r\Big|_1^{\frac{1}{\sin\theta}}$ $\qquad dr = \frac{-\cos\theta}{\sin^2\theta}\,d\theta$

$\phi = \frac{\pi}{2}$

$\frac{\sin\theta}{\cos\theta} = \frac{1}{2} = \tan\theta_b$



$\theta_b = 26.56°$

(iii) $r\Big|_{\sqrt{5}}^{0}, \ d\phi = 0, \ d\theta = 0$ $\qquad \sin\theta_b = 0.447$

$\phi = \pi/2 \quad \theta = \theta_b$ $\qquad \sin^2\theta_b = 0.200$

$\cos\theta_b = 0.894$

(iv) $r\Big|_0^1, \ d\theta = d\phi = 0$ $\qquad \cos^2\theta_b = 0.800$

$\theta = \frac{\pi}{2}, \ \phi = 0$

(ii) $\int \vec{v}\cdot d\vec{l} = \int v_r\, dl_r + \int v_\theta\, dl_\theta = \int r\cos^2\theta\, dr - \int (r\cos\theta\sin\theta)\, r\, d\theta$

$= \int \frac{\cos^2\theta}{\sin\theta}\left(\frac{-\cos\theta}{\sin^2\theta}\right) d\theta - \int \frac{\cos\theta\sin\theta}{\sin^2\theta}\, d\theta$

$\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta}{\sin^2\theta} + 1\cdot\frac{\sin^2\theta}{\sin^2\theta}\right) = \frac{\cos\theta}{\sin\theta}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)$

$\int \vec{v}\cdot d\vec{l} = \int \frac{\cos\theta}{\sin^3\theta}\, d\theta = \frac{1}{2\sin^2\theta}\Big|_{\pi/2}^{\theta_b} = \frac{1}{2}\left(\frac{1}{0.20} - \frac{1}{1} = 5 - 1 = 4\right) = \frac{4}{2} = 2$

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(iii) $\int \vec{v} \cdot d\vec{l} = \int v_r\,dr = \int r\cos^2\theta\,dr = \frac{r^2}{2}\cos^2\theta_b = \frac{0.8}{2} r^2 \Big|_{\sqrt{5}}^{0}$

$= 0.4\,(0-5)$

$= -2$

(iv) $\int \vec{v} \cdot d\vec{l} = \int v_r\,dr = \int r\cos^2\theta\,dr = \frac{r^2}{2}\cos^2\frac{\pi}{2} = 0$

$$\oint \vec{v} \cdot d\vec{l} = \underset{(i)}{\frac{3\pi}{2}} + \underset{(ii)}{2} - \underset{(iii)}{2} + \underset{(iv)}{0} = \frac{3\pi}{2}$$

To check with Stokes' theorem, I need $\nabla \times \vec{v}$ in spherical coordinates:

$(\nabla \times v)_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta\, v_\phi - \frac{\partial v_\theta}{\partial\phi}\right]$

$\frac{\partial v_\theta}{\partial\phi} = \frac{\partial}{\partial\phi}(+r\cos\theta\sin\theta) = 0$

$\frac{\partial}{\partial\theta}\sin\theta\, v_\phi = \frac{\partial}{\partial\theta}\sin\theta\, 3r = 3r\cos\theta$

$(\nabla \times v)_r = \frac{3r\cos\theta}{r\sin\theta} = 3\frac{\cos\theta}{\sin\theta}$

$(\nabla \times v)_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]$

$\frac{\partial}{\partial r}(r v_\phi) = \frac{\partial}{\partial r}(r \cdot 3r) = \frac{\partial}{\partial r}3r^2 = 6r$

$\frac{\partial v_r}{\partial\phi} = \frac{\partial}{\partial\phi} r\cos^2\theta = 0 \rightarrow (\nabla \times v)_\theta = \frac{1}{r}(-6r$

$(\nabla \times v)_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right] \quad \frac{\partial v_r}{\partial\theta} = \frac{\partial}{\partial\theta} r\cos^2\theta = -2r\cos\theta\sin\theta$

$\frac{\partial}{\partial r}(r v_\theta) = \frac{\partial}{\partial r}(r^2\cos\theta\sin\theta) = -2r\cos\theta\sin\theta \rightarrow (\nabla \times v)_\phi =$

1.56 continued: Stokes' theorem: $\oint \bar{v} \cdot d\bar{l} = \int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a}$

$(\bar{\nabla} \times \bar{v}) = 3 \frac{\cos\theta}{\sin\theta} \hat{r} - 6\hat{\theta} + 0\hat{\phi}$

Back face: $d\bar{a} = -r\, dr\, d\theta\ \hat{\phi}$, $(\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = 0$

Bottom: $d\bar{a} = -r \sin\theta\, dr\, d\phi\, \hat{\theta}$, $(\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = +6r \sin\theta\, dr\, d\phi$

$\theta = \pi/2$

$$\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = \int_{r=0}^{1} 6r\, dr \int_{0}^{\pi/2} d\phi$$

$$= \frac{6r^2}{2}\Big|_0^1 \phi\Big|_0^{\pi/2}$$

$$= 3 \cdot 1 \cdot \pi/2 = 3\pi/2 = \int \vec{v} \cdot d\vec{l}$$

Stokes' theorem checks.