Vector Calculus (Physical Systems) Midterm – Tues.6.Feb.2007

This is a CLOSED-BOOK exam to be taken in class. You may use one page of your notes. Please staple your notes to the exam when you hand it in.

This is designed as a half-hour quiz. You have 1 hour to do it.
SHOW YOUR WORK to receive full credit, and include units where appropriate
Please circle or underline your answers when appropriate, for clarity.
Express answers in *simplest exact form*.

(sign legibly) ZETA- SOLUTIONS

I affirm that I have worked this exam with WITHOUT using a calculator, text, HW, quizzes, computer, classmates, or other resources, except my one page of notes (attached).

1. Is each function below a vector or a scalar?

2 **(a)** $\mathbf{v} = x \sin y\, \hat{\mathbf{x}} + \cos y\, \hat{\mathbf{y}} + x y\, \hat{\mathbf{z}}$ VECTOR

(b) $T = e^{-5x} \sin 4y \cos 3z$ SCALAR

2. What is the definition of the ~~function~~ operator del or ∇ (in Cartesian coordinates)?

2 $$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

1 **3. (a) Can one find the Laplacian (∇^2) of a scalar or vector function?**

$\frac{\partial}{\partial x} e^{-5x} = -5e^{-5x}$, $\frac{\partial^2}{\partial x^2} e^{-5x} = 25e^{-5x}$

(b) Find the Laplacian of the appropriate function from (1) above.

$\frac{\partial}{\partial y} \sin 4y = 4\cos 4y$, $\frac{\partial^2}{\partial y^2} \sin 4y = -16 \sin 4y$

2 $$\nabla^2 T = \frac{\partial^2}{\partial x^2}T + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}.$$

$\frac{\partial}{\partial z}\cos 3z = -3\sin 3z$, $\frac{\partial^2}{\partial z^2}\cos 3z = -9\cos 3z$

$$\nabla^2 T = 25e^{-5x}(\sin 4y \cos 3z) - 16\sin 4y\,(e^{-5x}\cos 3z) - 9\cos 3z\,(e^{-5x} \cos 3z)$$
$$= (25 - 16 - 9)\,T = 0$$

1 **4 (a) Can one find the divergence and curl of a scalar or vector function?**

(b, c) Find the divergence and curl of the appropriate function from (1) above.

2 $$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y) + \frac{\partial}{\partial z}(xy)$$
$$= \sin y + (-\sin y) + 0 = 0$$

2 $$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\sin y & \cos y & xy \end{vmatrix} = \begin{matrix} \hat{x}\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}\cos y\right) \\ -\hat{y}\left(\frac{\partial}{\partial x}xy - \frac{\partial}{\partial z}x\sin y\right) \\ +\hat{z}\left(\frac{\partial}{\partial x}\cos y - \frac{\partial}{\partial y}x\sin y\right) \end{matrix}$$

12 $$\vec{\nabla}\times\vec{v} = \hat{x}(x-0) - \hat{y}(y-0) + \hat{z}(x\cos y) = x\hat{x} - y\hat{y} + x\cos y\,\hat{z}$$

5. The height of a hill (in meters) is given by $h = 32 - x^2 - 4y^2$.

1 **(a) Describe your reasoning and strategy for finding the top of the hill.**

$\vec{\nabla} h = 0$ at the top. Set $\frac{\partial h}{\partial x} = 0$ and $\frac{\partial h}{\partial y} = 0$

(No slope: flat) (and h is a decreasing function of x & y)

2 **(b) Find the coordinates at the top of the hill.**

$\frac{\partial h}{\partial x} = -2x = 0$ when $x = 0$ $\qquad$ $\frac{\partial h}{\partial y} = -8y = 0$ when $y = 0$

top of hill is at $(x,y) = (0,0)$

2 **(c) Describe your reasoning and strategy for finding the slope of the hill at any point.**

$\vec{\nabla} h = \hat{x}\frac{\partial h}{\partial x} + \hat{y}\frac{\partial h}{\partial y}$ = gradient or direction of greatest increase in slope of hill

$\vec{\nabla} h \cdot \bar{u}$ = slope in the direction $\bar{u}$.

3 **(d) If you start at (x,y) = (3,2) and in the direction $\hat{x} + \hat{y}$, are you going uphill or downhill?**
(e) How fast? (how steep?) $\qquad$ direction unit vector $\bar{u} = \frac{\hat{x}+\hat{y}}{\sqrt{1+1}} = \frac{\hat{x}+\hat{y}}{\sqrt{2}}$

$\vec{\nabla} h = -2x\hat{x} - 8y\hat{y}$

$\vec{\nabla} h(3,2) = -6\hat{x} - 16\hat{y}$

$\vec{\nabla} h(3,2) \cdot \bar{u} = \frac{-6-16}{\sqrt{2}} = \frac{-22}{\sqrt{2}} = \frac{-2}{\sqrt{2}}\,11 = -11\sqrt{2}$ meters/m

DOWNHILL