Vector Calculus HW #3 due Tues 6 Feb 2007

E/Z

Problem 1.31 Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0,0,0)$, $\mathbf{b} = (1,1,1)$, and the three paths in Fig. 1.28:

(a) $(0,0,0) \xrightarrow{(i)} (1,0,0) \xrightarrow{(ii)} (1,1,0) \xrightarrow{(iii)} (1,1,1)$;

(b) $(0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,1) \rightarrow (1,1,1)$;

(c) the parabolic path $z = x^2$; $y = x$.

$$\int_{a \text{ path}}^{b} (\nabla T)\cdot d\bar{l} = T(b) - T(a)$$

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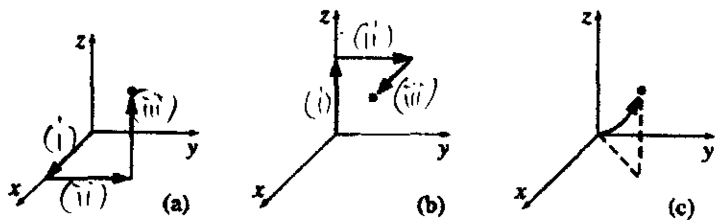


(a) (b) (c)

$T(b) = 1^2 + 4\cdot1\cdot1 + 2\cdot1\cdot1^3 = 1+4+2 = 7$, $T(a) = 0 \rightarrow \underline{T(b) - T(a) = 7}$

$$\nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$
$$= \hat{x}(2x+4y) + \hat{y}(4x+2z^3) + \hat{z}(6yz^2)$$

ⓐ (i) $dx \neq 0$, $x\big|_0^1$, $y = z = 0$ / (ii) $dy \neq 0$, $x=1$, $z=0$, $y\big|_0^1$ / (iii) $dz \neq 0$, $x=y=1$, $z\big|_0^1$

$$\int (\nabla T)\cdot d\bar{l} = \int_{(i)} \nabla T_x\, dx + \int_{(ii)} \nabla T_y\, dy + \int_{(iii)} \nabla T_z\, dz$$

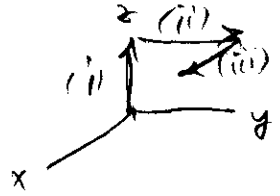
$$= \int_{x=0}^{1} (2x + \cancel{4y}^{\,0})\,dx + \int_{y=0}^{1} (4x + \cancel{2z^3}^{\,0})\,dy + \int_{z=0}^{1} (6yz^2)\,dz$$

$$= \int_0^1 2x\,dx \quad + \int 4\,dy \quad + \int 6z^2\,dz$$

$$= \frac{2x^2}{2}\Big|_0^1 \quad + 4y\Big|_0^1 \quad + \frac{6z^3}{3}\Big|_0^1$$

$$= 1 - 0 \quad + 4(1-0) \quad + 2(1-0)$$

$$= 1 \quad + 4 \quad + 2 \quad = 7 \checkmark$$



(b) (i) $dz \neq 0, x = y = 0$ / (ii) $dy \neq 0, x = 0, z = 1$ / (iii) $dx \neq 0, y = z = 1$

$$\int(\bar{\nabla}T)\cdot d\bar{l} = \underset{(iii)}{\int}\nabla T_x\,dx + \underset{(ii)}{\int}\nabla T_y\,dy + \underset{(i)}{\int}\nabla T_z\,dz$$

$$= \underset{(iii)}{\int}(2x+4y)\,dx + \underset{(ii)}{\int}(4x+2z^3)\,dy + \underset{(i)}{\int}(6yz^2)\,dz$$

$$= \int_{x=0}^{1}(2x+4)\,dx + \int_{y=0}^{1}(4\cdot 0+2)\,dy + \int_{z=0}^{1} 0\,dz$$

$$= \frac{2x^2}{2} + 4x\Big|_0^1 + 2y\Big|_0^1 + 0$$

$$= 1 + 4 - 0 + 2 - 0 = 7 \checkmark$$

(c) Parabolic path $z = x^2,\ y = x \rightarrow z = y^2,\ y = \sqrt{z}$

$$\int(\bar{\nabla}T)\cdot d\bar{l} = \int(2x+4y)\,dx + \int(4x+2z^3)\,dy + \int(6yz^2)\,dz$$

$$= \int_{x=0}^{1}(2x+4x)\,dx + \int_{y=0}^{1}(4y+2y^6)\,dy + \int_{z=0}^{1}(6z^{5/2})\,dz$$

$$= \frac{6x^2}{2}\Big|_0^1 + \frac{4y^2}{2} + \frac{2y^7}{7}\Big|_0^1 + \frac{6z^{7/2}}{7/2}\Big|_0^1$$

$$= 3\cdot 1 + 2\cdot 1 + \frac{2}{7}\cdot 1 + \frac{12}{7}\cdot 1$$

$$= 3 + 2 + 2\left(\frac{1}{7} + \frac{6}{7}\right)$$

$$= 5 + 2 = 7 \checkmark$$

Problem 1.33 Test Stokes' theorem for the function $\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}}$, using the triangular shaded area of Fig. 1.34.

$$\underbrace{\int(\bar{\nabla}\times\bar{v})\cdot d\bar{a}}_{\text{surface}} = \underbrace{\oint \bar{v}\cdot d\bar{l}}_{\text{path}}$$

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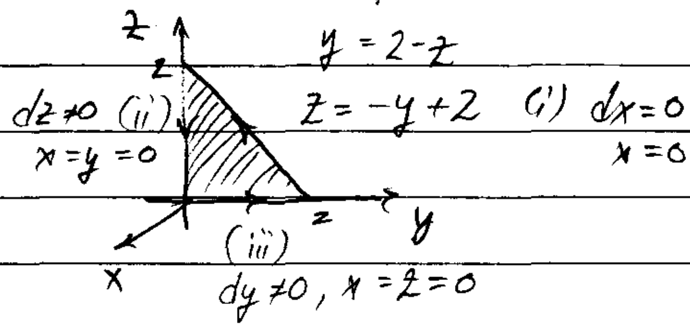

$$\oint \vec{v}\cdot d\vec{l} = \underbrace{\int v_x\,dx}_{\text{nowhere}} + \underbrace{\int v_y\,dy}_{\text{path (i) and (iii)}} + \underbrace{\int v_z\,dz}_{\text{path (i) and (ii)}}$$

$$= \int_{(iii)} 2y\cancelto{0}{z}\,dy + \int_{(ii)} 3z\cancelto{0}{x}\,dz + \int_{(i)} 2yz\,dy$$

$$= \int_{i} 2y(-y+2)\,dy = 2\int_{y=2}^{0} (2y - y^2)\,dy$$

$$= 2\left(\frac{2y^2}{2} - \frac{y^3}{3}\right)\Big|_2^0 = 2\left(0 - 4 - \left[0 - \frac{8}{3}\right]\right)$$

$$= 2\left(\frac{8}{3} - \frac{12}{3}\right) = \frac{2}{3}(-4) = -\frac{8}{3}$$

$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3xz \end{vmatrix} = \begin{matrix} \hat{x}\left(\frac{\partial}{\partial y}3xz - \frac{\partial}{\partial z}2yz\right) \\ -\,\hat{y}\left(\frac{\partial}{\partial x}3xz - \frac{\partial}{\partial z}xy\right) \\ +\,\hat{z}\left(\frac{\partial}{\partial x}2yz - \frac{\partial}{\partial y}xy\right) \end{matrix}$$

$$\bar{\nabla}\times\bar{v} = \hat{x}(0 - 2y) - \hat{y}(3z - 0) + \hat{z}(0 - x) = -2y\hat{x} - 3z\hat{y} - x\hat{z}$$

$$d\vec{a} = \hat{x}\,dy\,dz$$

$$\oint(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint(\nabla\times v)_x\,dy\,dz = \int_z\int_y -2y\,dy\,dz = \int_z\left(-\frac{2y^2}{2}\right)\Big|_{2-z}^{0} dz$$

$$= \int_z -\left(0 - [2-z]^2\right)dz = \int_{z=2}^{0}(4 - 4z + z^2)\,dz = 4z - \frac{4z^2}{2} + \frac{z^3}{3}\Big|_2^0$$

$$= 4(0-2) - 2(0-4) + \left(0 - \frac{8}{3}\right) = -8 + 8 - \frac{8}{3} = -\frac{8}{3}$$