3.9 First, do

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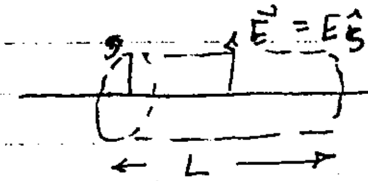
Problem 2.22 Find the potential a distance r from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.



$\lambda = \frac{dq}{dl}$

Flux through Gaussian cylinder $\oint \bar{E}\cdot d\bar{a} = \frac{q}{\epsilon_0}$

$E(r)\, 2\pi s\, L = \lambda L/\epsilon_0$

$\bar{E}(s) = \frac{\lambda}{2\pi s\epsilon_0}\hat{s}$

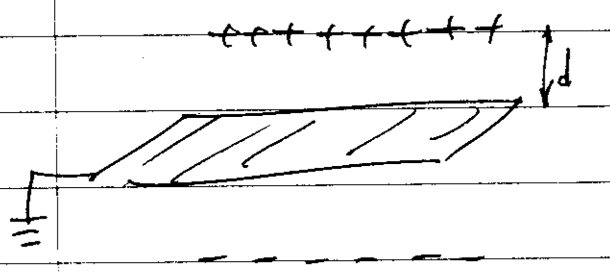
$V(r) = -\int_a^s \bar{E}\cdot d\bar{l} = -\frac{\lambda}{2\pi\epsilon_0}\int_a^s \frac{ds}{s} = -\frac{\lambda}{2\pi\epsilon_0}(\ln s - \ln a)$

BC: Fix Reference point at a: $V(a) = 0$

Can't choose $V(\infty) = 0$ because λ extends to ∞!

Check: $-\nabla V = \frac{\partial}{\partial s}\left(+\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s}{a}\right)\right)\hat{s} = -\frac{\lambda}{2\pi\epsilon_0 s}\hat{s} = \bar{E}$ ✓

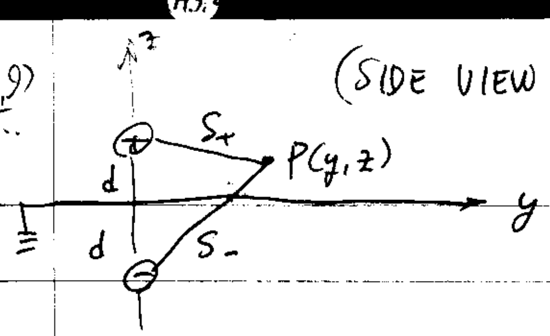
3.9 Now put this line of charge above a grounded conducting plane

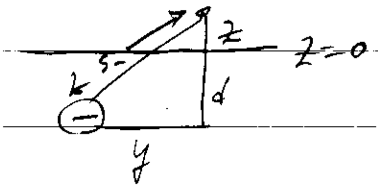


Guess that the induced charge on the plane has the same field as a line of ⊖ charge a distance d BELOW the sheet.

By symmetry, V=0 at sheet ∴ this solution works.

3,9) (SIDE VIEW)





$S_+^2 = (z-d)^2 + y^2$ $\qquad$ $S_-^2 = (z+d)^2 + y^2$

We found $V_+(S_+) = \dfrac{-\lambda}{2\pi\epsilon_0} \ln\left(\dfrac{S_+}{a}\right)$. Also $V_-(S_-) = \dfrac{-\lambda}{2\pi\epsilon_0} \ln\left(\dfrac{S_-}{a}\right)$

Potential at $P(y,z) = V_+(S_+) - V_-(S_-)$

$$= \frac{-\lambda}{2\pi\epsilon_0}\left[(\ln S_+ - \ln a) - (\ln S_- - \ln a)\right]$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\left[\ln S_+ - \ln S_-\right] = +\frac{\lambda}{2\pi\epsilon_0}\left[\ln S_- - \ln S_+\right] = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{S_-}{S_+}$$

NB: $\ln x = \ln(x^2)^{\frac{1}{2}} = \frac{1}{2}\ln x^2$

So $\ln \dfrac{S_-}{S_+} = \dfrac{1}{2}\ln\left(\dfrac{S_-^2}{S_+^2}\right)$

and $V(P) = \dfrac{\lambda}{2\pi\epsilon_0}\,\dfrac{1}{2}\ln\left[\dfrac{(z+d)^2+y^2}{(z-d)^2+y^2}\right] = \dfrac{\lambda}{4\pi\epsilon_0}\ln\left(\dfrac{f(z)}{g(z)}\right)$

(b) Find charge density σ induced on the conducting plane:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{-\lambda}{4\pi}\frac{\partial}{\partial z}\ln\left(\frac{f(z)}{g(z)}\right)$$

$$\frac{\partial}{\partial z}\ln\frac{f(z)}{g(z)} = \frac{1}{f/g}\,\frac{\partial}{\partial z}\left(\frac{f}{g}\right) = \frac{g}{f}\,\frac{1}{g^2}(gf' - fg') = \frac{1}{fg}(gf' - fg')$$

$$f' = \frac{\partial}{\partial z}\left[(z+d)^2 + y^2\right] = 2(z+d), \quad g' = \frac{\partial}{\partial z}\left[(z-d)^2 + y^2\right] = 2(z-d)$$