Problem 1.13

$\boldsymbol{\mathscr{r}} = (x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}; \quad \mathscr{r} = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}.$

(a) $\nabla(\mathscr{r}^2) = \frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}()\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}()\,\hat{\mathbf{z}} = 2(x-x')\,\hat{\mathbf{x}} + 2(y-y')\,\hat{\mathbf{y}} + 2(z-z')\,\hat{\mathbf{z}} = 2\boldsymbol{\mathscr{r}}.$

(b) $\nabla(\frac{1}{\mathscr{r}}) = \frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]^{-\frac{1}{2}}\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}()^{-\frac{1}{2}}\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}()^{-\frac{1}{2}}\,\hat{\mathbf{z}}$
$= -\frac{1}{2}()^{-\frac{3}{2}}2(x-x')\,\hat{\mathbf{x}} - \frac{1}{2}()^{-\frac{3}{2}}2(y-y')\,\hat{\mathbf{y}} - \frac{1}{2}()^{-\frac{3}{2}}2(z-z')\,\hat{\mathbf{z}}$
$= -()^{-\frac{3}{2}}[(x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}] = -(1/\mathscr{r}^3)\boldsymbol{\mathscr{r}} = -(1/\mathscr{r}^2)\hat{\boldsymbol{\mathscr{r}}}.$

(c) $\frac{\partial}{\partial x}(\mathscr{r}^n) = n\mathscr{r}^{n-1}\frac{\partial \mathscr{r}}{\partial x} = n\mathscr{r}^{n-1}(\frac{1}{2}\frac{1}{\mathscr{r}}2\mathscr{r}_x) = n\mathscr{r}^{n-1}\hat{\mathscr{r}}_x$, so $\boxed{\nabla(\mathscr{r}^n) = n\mathscr{r}^{n-1}\hat{\boldsymbol{\mathscr{r}}}.}$

Problem 1.14

$\bar{y} = +y\cos\phi + z\sin\phi$; multiply by $\sin\phi$: $\bar{y}\sin\phi = +y\sin\phi\cos\phi + z\sin^2\phi$.
$\bar{z} = -y\sin\phi + z\cos\phi$; multiply by $\cos\phi$: $\bar{z}\cos\phi = -y\sin\phi\cos\phi + z\cos^2\phi$.
Add: $\bar{y}\sin\phi + \bar{z}\cos\phi = z(\sin^2\phi + \cos^2\phi) = z$. Likewise, $\bar{y}\cos\phi - \bar{z}\sin\phi = y$.
So $\frac{\partial y}{\partial \bar{y}} = \cos\phi$; $\frac{\partial y}{\partial \bar{z}} = -\sin\phi$; $\frac{\partial z}{\partial \bar{y}} = \sin\phi$; $\frac{\partial z}{\partial \bar{z}} = \cos\phi$. Therefore

$$\left.\begin{array}{l} \overline{(\nabla f)}_y = \frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}} = +\cos\phi(\nabla f)_y + \sin\phi(\nabla f)_z \\ \overline{(\nabla f)}_z = \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{z}} = -\sin\phi(\nabla f)_y + \cos\phi(\nabla f)_z \end{array}\right\} \text{So } \nabla f \text{ transforms as a vector.} \quad \text{qed}$$

Problem 1.15

$(a)\,\nabla\cdot\mathbf{v}_a = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) = 2x + 0 - 2x = 0.$

$(b)\,\nabla\cdot\mathbf{v}_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3xz) = y + 2z + 3x.$

$(c)\,\nabla\cdot\mathbf{v}_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz) = 0 + (2x) + (2y) = 2(x+y).$

Problem 1.16

$\nabla\cdot\mathbf{v} = \frac{\partial}{\partial x}(\frac{x}{r^3}) + \frac{\partial}{\partial y}(\frac{y}{r^3}) + \frac{\partial}{\partial z}(\frac{z}{r^3}) = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$
$= ()^{-\frac{3}{2}} + x(-3/2)()^{-\frac{5}{2}}2x + ()^{-\frac{3}{2}} + y(-3/2)()^{-\frac{5}{2}}2y + ()^{-\frac{3}{2}} + z(-3/2)()^{-\frac{5}{2}}2z$
$= 3r^{-3} - 3r^{-5}(x^2+y^2+z^2) = 3r^{-3} - 3r^{-3} = 0.$

This conclusion is surprising, because, from the diagram, this vector field is obviously diverging away from the origin. How, then, can $\nabla\cdot\mathbf{v} = 0$? The answer is that $\nabla\cdot\mathbf{v} = 0$ everywhere *except* at the origin, but at the origin our calculation is no good, since $r = 0$, and the expression for $\mathbf{v}$ blows up. In fact, $\nabla\cdot\mathbf{v}$ is *infinite* at that one point, and zero elsewhere, as we shall see in Sect. 1.5.

Problem 1.17

$\bar{v}_y = \cos\phi\, v_y + \sin\phi\, v_z$; $\bar{v}_z = -\sin\phi\, v_y + \cos\phi\, v_z$.

$\frac{\partial \bar{v}_y}{\partial \bar{y}} = \frac{\partial v_y}{\partial \bar{y}}\cos\phi + \frac{\partial v_z}{\partial \bar{y}}\sin\phi = \left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \bar{y}}\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial \bar{y}}\right)\sin\phi$. Use result in Prob. 1.14:
$= \left(\frac{\partial v_y}{\partial y}\cos\phi + \frac{\partial v_y}{\partial z}\sin\phi\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi\right)\sin\phi.$

$\frac{\partial \bar{v}_z}{\partial \bar{z}} = -\frac{\partial v_y}{\partial \bar{z}}\sin\phi + \frac{\partial v_z}{\partial \bar{z}}\cos\phi = -\left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \bar{z}}\right)\sin\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial \bar{z}}\right)\cos\phi$
$= -\left(-\frac{\partial v_y}{\partial y}\sin\phi + \frac{\partial v_y}{\partial z}\cos\phi\right)\sin\phi + \left(-\frac{\partial v_z}{\partial y}\sin\phi + \frac{\partial v_z}{\partial z}\cos\phi\right)\cos\phi.$ So

$\frac{\partial \bar{v}_y}{\partial \bar{y}} + \frac{\partial \bar{v}_z}{\partial \bar{z}} = \frac{\partial v_y}{\partial y}\cos^2\phi + \frac{\partial v_y}{\partial z}\sin\phi\cos\phi + \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\sin^2\phi + \frac{\partial v_y}{\partial y}\sin^2\phi - \frac{\partial v_y}{\partial z}\sin\phi\cos\phi$