EM HW #2 - Physical Systems - due 30 Jan 2007

E/Z

2.32(a,b) ✓ 2.40 ✓ 2.36(a-c) 2.45 (one way)

Problem 2.36 Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R (Fig. 2.49). At the center of each cavity a point charge is placed—call these charges q_a and q_b.

(a) Find the surface charges σ_a, σ_b, and σ_R.

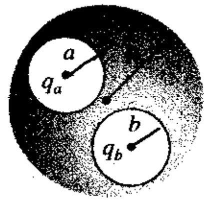
(b) What is the field outside the conductor?

(c) What is the field within each cavity?

(d) What is the force on q_a and q_b?

(e) Which of these answers would change if a third charge, q_c, were brought near the conductor?

(a) A cavity containing charge q attracts charge $-q$ to the cavity surface, so $E=0$ just outside the cavity surface.



$\sigma_a = \frac{-q_a}{4\pi a^2}$ and $\sigma_b = \frac{-q_b}{4\pi b^2}$

Since the conducting sphere is neutral, the same amount of (opposite) charge must be driven to its outside surface: $\boxed{Q = q_a + q_b}$, $\sigma_R = \frac{Q}{4\pi R^2}$

(b) The field inside a conductor is ALWAYS ZERO in electrostatics. If $E \neq 0$, that will make charges move. They will redistribute themselves until $E=0$.

OUTSIDE the conductor, the field is the same as that due to a point charge Q at the center: $E(r>R) = \frac{kQ}{4\pi r^2}$

(c) Within each cavity, the field is that due to the charge in the cavity only.

$E(r<a) = \frac{kq_a}{4\pi r^2}$ $E(r<b) = \frac{kq_b}{4\pi r^2}$

The intervening conducting material SHIELDS each cavity from fields due to all other charges or, equivalently, the surface charge around each cavity shields them.

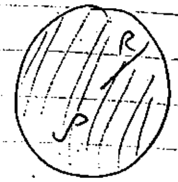
(d) Since E_a is due to q_a only, $F_a = 0$.
Same for b.

(e) An outside charge could induce more (or fewer) charges to move to the sphere's outer surface, but nothing INSIDE would change.

$q = \frac{4}{3}\pi R^3 \rho$

2.32

Problem ~~2.33~~ Find the energy stored in a uniformly charged solid sphere of radius R and total charge q. Do it three different ways:
(a) Use equation (2.37). You found the potential in Problem 2.21.
(b) Use equation (2.39). Don't forget to integrate over *all space*.
(c) Use equation (2.38). Take a spherical volume of radius a. Notice what happens as $a \to \infty$.



(a) $W = \frac{1}{2}\int \rho V \, d\tau$, integrated over the charge distribution

(2.43) In problem 2.21, we found $V_{in} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) = \frac{\rho}{6\epsilon_0}(3R^2 - r^2)$ ✓

and $V_{out} = \frac{\rho R^3}{3\epsilon_0 r} = \frac{q}{4\pi\epsilon_0 r}$

$$W = \frac{1}{2}\int_0^R \rho V_{in} \, 4\pi r^2 dr = \frac{1}{2}\left(\frac{3q}{4\pi R^3}\right) 4\pi \int \frac{q}{4\pi\epsilon_0}\left(\frac{1}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) r^2 dr$$

$$= \frac{3q^2}{2R^3 \, 2R} \frac{1}{4\pi\epsilon_0} \int_0^R \left(3r^2 - \frac{r^4}{R^2}\right) dr$$

$$= \frac{3q^2}{4R^4} \frac{1}{4\pi\epsilon_0} \left(\frac{3r^3}{3} - \frac{r^5}{5R^2}\right)\Big|_0^R = \frac{3q^2}{4\pi\epsilon_0}\frac{1}{4R^4}\left(R^3 - \frac{R^5}{5R^2}\right)$$

$$\frac{4}{5}R^3$$

$$W = \frac{q^2}{4\pi\epsilon_0}\frac{3}{5R} = \frac{4\pi\rho^2}{15\epsilon_0} R^5$$

(b) Another way to find the energy stored - done in class:

(2.45) $W = \frac{\epsilon_0}{2}\int E^2 d\tau$, integrated over all space (over field)

since total charge $q = \frac{4}{3}\rho\pi R^3$, the charge inside a radius $r < R$ is

$$q(r) = \frac{4}{3}\rho\pi r^3 = q\frac{r^3}{R^3}$$

By Gauss' law, $E_{in}(r<R) = \frac{qr}{4\pi\epsilon_0 R^3}$ and $E_{out}(r>R) = \frac{q}{4\pi\epsilon_0 r^2}$

$$E_{in}(r<R) = \frac{\frac{4}{3}\rho\pi r^3 \, r}{4\pi\epsilon_0 R^3} = \frac{\rho r^4}{3\epsilon_0 R^3}$$

$$E_{out}(r>R) = \frac{\frac{4}{3}\rho\pi r^3}{4\pi\epsilon_0 r^2} = \frac{\rho r}{3\epsilon_0}$$

2.32.b continued $W = \frac{\epsilon_0}{2}\int E^2\, d\tau = \frac{\epsilon_0}{2} 4\pi \left[\int_{r=0}^{R} E_{in}^2\, r^2 dr + \int_{r=R}^{\infty} E_{out}^2\, r^2 dr \right]$

$$\int_{r=0}^{R} E_{in}^2\, r^2 dr = \left(\frac{q}{4\pi\epsilon_0 R^3}\right)^2 \int_0^R r^2 r^2\, dr = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{R^5}{5R^6}$$

$$\int_{r=R}^{\infty} E_{out}^2\, r^2 dr = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \int \left(\frac{1}{r^2}\right)^2 r^2\, dr = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \left(-\frac{1}{\infty} + \frac{1}{R}\right)$$

So $W = \frac{\epsilon_0}{2} \frac{q^2}{4\pi\epsilon_0{}^2} \left(\frac{1}{R} + \frac{1}{5R} = \frac{6}{5R} \right) = \frac{q^2}{4\pi\epsilon_0} \frac{3}{5R}$

(c) Yet another way to find energy stored ~~(2.38)~~ (2.44)

$W = \frac{\epsilon_0}{2} \left(\oint_{\text{surface}} V \vec{E} \cdot d\vec{a} + \int_{\text{volume}} E^2\, d\tau \right)$ can be calculated at any radius a outside R.

$\oint_{\text{surface at } r=a>R} V \vec{E} \cdot d\vec{a} = V_{out} E_{out} 4\pi a^2$ since V and E are constant at a

$$= \left(\frac{q}{4\pi\epsilon_0 a}\right)\left(\frac{q\, a}{4\pi\epsilon_0 a^3}\right) 4\pi a^2 = \frac{q^2}{4\pi\epsilon_0{}^2 a}$$

NOTICE THAT SURFACE CONTRIBUTION VANISHES as $a \to \infty$.

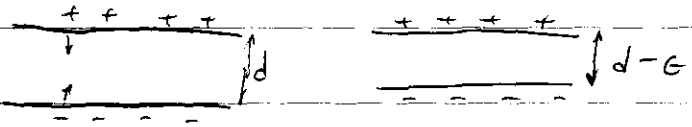
$$\int_{\text{volume at } a} E^2\, d\tau = \int_{r=0}^{R} E_{in}^2\, 4\pi r^2 dr + \int_{r=R}^{a} E_{out}^2\, 4\pi r^2 dr$$

$$= \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{5R} + \left(\frac{q}{4\pi\epsilon_0}\right)^2 \left(-\frac{1}{a} + \frac{1}{R}\right) 4\pi$$

$$W = \frac{\epsilon_0}{2} \frac{q^2}{4\pi\epsilon_0{}^2} \left[\underbrace{\cancel{\frac{1}{a}}}_{\text{surface}} + \underbrace{\frac{1}{5R}}_{\text{vol inside}} \underbrace{- \cancel{\frac{1}{a}} + \frac{1}{R}}_{\text{volume outside}} \right] = \frac{q^2}{4\pi\epsilon_0} \frac{3}{5R}$$

$\frac{2.40}{106}$ Suppose the plates of a parallel plate capacitor move closer together by a small distance ϵ as a result of their mutual attraction.

Plates have area A.



(a) Eqn $\frac{(2.52)}{103}$ Pressure on surface $P = \frac{\epsilon_0}{2} E^2 = \frac{\text{Energy}}{\text{volume}} = \frac{\text{force}}{\text{area}}$

Work done = force × distance = pressure · area · distance

$= \left(\frac{\epsilon_0}{2} E^2\right) \quad A \quad \epsilon$

(b) Energy lost by field = energy density × volume $\frac{(2.46)}{96}$

$= \frac{\epsilon_0 E^2}{2} \quad (A \cdot \epsilon)$

= work done in moving plates

2.45 / 107 A sphere of radius R carries a charge density $\rho(r) = kr$ where k = constant. (NOT $k = \frac{1}{4\pi\epsilon_0}$!) Find the energy of the configuration.

First find the charge enclosed by any radius.

$$\rho = \frac{dq}{d\text{volume}} \Rightarrow \int dq = \int \rho \, d\tau = \int_0^r kr \, 4\pi r^2 \, dr \qquad \int r^3 dr = \frac{r^4}{4}$$

$$\text{INSIDE } q(r<R) = \frac{4\pi k r^4}{4} = \pi k r^4$$

Total charge enclosed $Q = \pi k R^4$

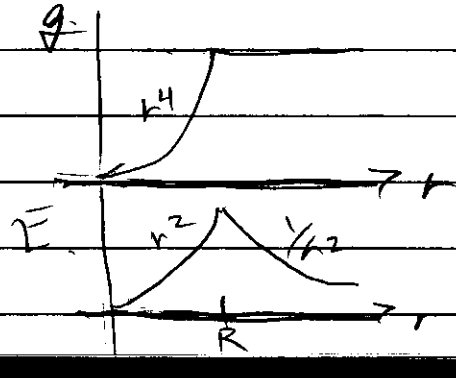
Now use Gauss's law to find the field everywhere (this is easier than finding $V(q)$ or $V(\rho)$, thanks to the symmetry)

$$\oint E \cdot dA = \frac{q(r)}{\epsilon_0} = E \cdot 4\pi r^2 \rightarrow \text{INSIDE } E(r<R) = \frac{\pi k r^4}{4\pi\epsilon_0 r^2} = \frac{kr^2}{4\epsilon_0}$$

$$E(R) = \frac{kR^2}{4\epsilon_0}$$

$$\text{outside } E(r>R) = \frac{Q_{tot}}{4\pi\epsilon_0 r^2} = \frac{\pi k R^4}{4\pi\epsilon_0 r^2} = \frac{kR^4}{4\epsilon_0 r^2}$$

sketch $q(r)$ & $E(r)$



(2.45) continued ... One way to find energy: $W = \frac{1}{2}\int \rho V d\tau$

Now, Starting from the outside, find $V(r)$ everywhere

OUTSIDE: $V(r>R) = -\int_{\infty}^{r} E_{out}\, dr = \frac{-kR^4}{4\epsilon_0}\int_{\infty}^{r}\frac{1}{r^2}\,dr = \frac{+kR^4}{4\epsilon_0}\left.\frac{1}{r}\right|_{\infty}^{r}$

$$V(r>R) = \frac{kR^4}{4\epsilon_0 r}$$

At the surface: $V(R) = \frac{kR^4}{4\epsilon_0 R} = kR^3/4\epsilon_0$

INSIDE: $V(r<R) = -\int_{\infty}^{R} E(r\geq R)\,dr - \int_{R}^{r} E_{in}\,dr = kR^3/4\epsilon_0 - \int_{R}^{r} kr^2/4\epsilon_0\,dr$

$$= \frac{kR^3}{4\epsilon_0} - \frac{k}{4\epsilon_0}\left.\frac{r^3}{3}\right|_{R}^{r} = \frac{k}{4\epsilon_0}\left[\underbrace{R^3 - \left(\frac{r^3}{3} - \frac{R^3}{3}\right)}_{\frac{4R^3}{3} - \frac{r^3}{3}}\right]$$

$$V(r<R) = \frac{k}{3\epsilon_0}\left[R^3 - \frac{r^3}{4}\right]$$

Finally, $W = \frac{1}{2}\int_0^R \rho V_{IN}\, d\tau$ where $\rho = kr$ and $d\tau = 4\pi r^2 dr$

$$= \frac{1}{2}\int kr \frac{k}{3\epsilon_0}\left[R^3 - \frac{r^3}{4}\right] 4\pi r^2\, dr$$

$$= 2\pi \frac{k^2}{3\epsilon_0}\int \left(R^3 r^3 - \frac{1}{4}r^6\right) dr =$$

$$= 2\pi\, k^2/3\epsilon_0 \left(R^3 \frac{r^4}{4} - \frac{1}{4\cdot 7}r^7\right)\Big|_0^R$$

$$= \frac{2\pi}{4}\frac{k^2}{3\epsilon_0}\left(R^7 - \frac{1}{7}R^7\right) \Rightarrow \left(\frac{6}{7}R^7\right)$$

$$= \frac{\pi}{2\cdot 3}\frac{k^2}{\epsilon_0}\left(\frac{6}{7}R^7\right) = \frac{\pi}{7}\frac{k^2 R^7}{\epsilon_0}$$

This is the energy stored in the charge distribution

Alternative: find the energy stored in the field:

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int_0^R E_{in}^2 d\tau + \frac{\epsilon_0}{2}\int_R^\infty E_{out}^2 d\tau$$

$$E_{in} = \frac{k}{4\epsilon_0} r^2 \qquad E_{out} = \frac{kR^4}{4\epsilon_0 r^2} \qquad d\tau = 4\pi r^2 dr$$

$$I_{in} = \int_0^R E_{in}^2 d\tau = \left(\frac{k}{4\epsilon_0}\right)^2 \int r^4 4\pi r^2 dr = 4\pi \frac{k^2}{4 \cdot 4\epsilon_0^2}\int r^6 dr$$

$$= \frac{\pi k^2}{4\epsilon_0^2} \frac{r^7}{7}\Big|_0^R = \frac{\pi k^2}{4\epsilon_0^2} R^7/7$$

$$I_{out} = \int_R^\infty E_{out}^2 d\tau = \left(\frac{kR^4}{4\epsilon_0}\right)^2 \int \frac{1}{r^4} 4\pi r^2 dr = \frac{k^2 R^8}{4 \cdot 4\epsilon_0^2} 4\pi \int r^{-2} dr$$

$$= \frac{\pi k^2 R^8}{4\epsilon_0^2}\left(-\frac{1}{r}\right)\Big|_R^\infty = -\frac{\pi k^2 R^8}{4\epsilon_0^2}\left(0 - \frac{1}{R}\right)$$

$$= \pi k^2 R^7 / 4\epsilon_0^2$$

$$I_{in} + I_{out} = \frac{\pi k^2}{4\epsilon_0^2}\left(\frac{R^7}{7} + R^7\right) \rightarrow \left(\frac{8R^7}{7}\right)$$

$$W = \frac{\epsilon_0}{2}(I_{in} + I_{out}) = \frac{\epsilon_0}{2}\frac{\pi k^2}{4\epsilon_0^2}\left(\frac{8R^7}{7}\right) = \frac{\pi k^2 R^7}{\epsilon_0 7}$$