#6a   Ch 10 #1, 2, 3, 5, 6

Griffiths EM - Physical Systems
Spring week 6
Tues 9 May 08
E/Z

Problem 10.1 Show that the differential equations for V and $\mathbf{A}$ (Eqs. 10.4 and 10.5) can be written in the more symmetrical form

$$\left.\begin{aligned} \Box^2 V + \frac{\partial L}{\partial t} &= -\frac{1}{\epsilon_0}\rho, \quad ① \\ \Box^2 \mathbf{A} - \nabla L &= -\mu_0 \mathbf{J}, \quad ② \end{aligned}\right\} \quad (1$$

where

$$ⓐ \quad \Box^2 \equiv \nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \quad \text{and} \quad L \equiv \nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}. \quad ⓑ$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{1}{\epsilon_0}\rho; \qquad (10.4)$$

this replaces Poisson's equation (to which it reduces in the static case). Putting Eqs. 10.2 and 10.3 into (iv) yields

$$\nabla\times(\nabla\times\mathbf{A}) = \mu_0\mathbf{J} - \mu_0\epsilon_0\nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2},$$

or, using the vector identity $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$, and rearranging the terms a bit:

$$\left(\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \qquad (10.5)$$

Equations 10.4 and 10.5 contain all the information in Maxwell's equations.

$$① \quad \Box^2 V + \frac{\partial L}{\partial t} = \underbrace{\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)}_{ⓐ} V + \frac{\partial}{\partial t}\underbrace{\left(\nabla\cdot A + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right)}_{ⓑ}$$

$$= \nabla^2 V - \cancel{\mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2}} + \frac{\partial}{\partial t}\nabla\cdot A + \cancel{\mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2}}$$

$$= \nabla^2 V + \frac{\partial}{\partial t}\nabla\cdot A = -\frac{\rho}{\epsilon_0} \quad (10.4) \checkmark$$

$$② \quad \Box^2 A - \nabla L = \underbrace{\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)}_{ⓐ} A - \nabla\underbrace{\left[\nabla\cdot A + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right]}_{ⓑ}$$

$$= \nabla^2 A - \mu_0\epsilon_0\frac{\partial^2 A}{\partial t^2} - \nabla(\nabla\cdot A) + \mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$$= -\mu_0 J \quad (10.5) \checkmark$$

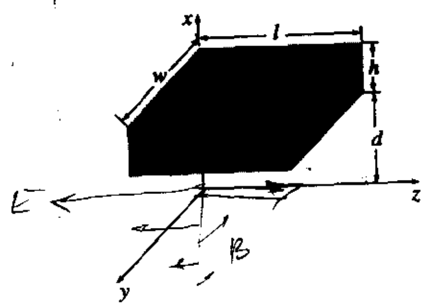
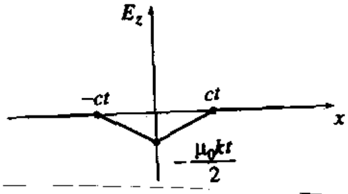
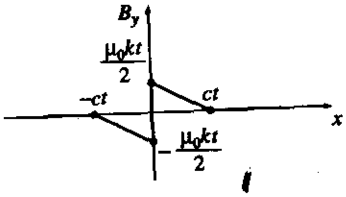


Figure 10.2

The fields spread out up & down along the x axis, with $\vec{E}$ pointing in the z direction, B pointing in the y dir.





$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\,\hat{\mathbf{z}},$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\,\hat{\mathbf{y}} = \pm\frac{\mu_0 k}{2c}(ct - |x|)\,\hat{\mathbf{y}},$$

$$B^2 = \frac{E^2}{c^2}$$

Problem 10.2 For the configuration in Ex. 10.1, consider a rectangular box of length l, width w, and height h, situated a distance d above the yz plane (Fig. 10.2).

(a) Find the energy in the box at time $t_1 = d/c$, and at $t_2 = (d+h)/c$.

(b) Find the Poynting vector, and determine the energy per unit time flowing into the box during the interval $t_1 < t < t_2$.

(c) Integrate the result in (b) from t_1 to t_2 and confirm that the increase in energy (part (a)) equals the net influx.

(a) Energy density $= u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} = \frac{\epsilon_0}{2}E^2 + \frac{E^2}{2\mu_0 c^2} = \epsilon_0 E^2$

At $t_1 = \frac{d}{c}$, the electric field has barely reached $x_{max} = ct_1 = \frac{cd}{c} = d$, so $E(x>d) = 0$.

Similarly, the magnetic field hasn't reached the box yet either, so <u>Energy in box $(t_1) = 0 = W_1$</u>

At $t_2 = \frac{d+h}{c}$, $x_{max} = ct_2 = d+h$, so E and B fields fill the box from $x = d$ to $(d+h)$.

$E_2(t_2) = -\frac{\mu_0 k}{2}(d+h-x)$. Volume $d\tau = l \cdot w \cdot dx$

Energy $= W_2 = \int u \cdot d\tau = \int \epsilon_0 E^2\, l \cdot w \cdot dx$

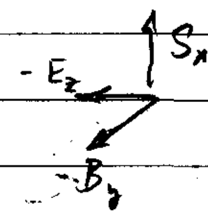
$= \epsilon_0 \left(\frac{\mu_0 k}{2}\right)^2 l \cdot w \int_d^{d+h} (d+h-x)^2\, dx$

$$W_2 = \epsilon_0 l w \left(\frac{\mu_0 k}{2}\right)^2 \left[-\frac{(d+h-x)^3}{3}\right]\Bigg|_d^{d+h}$$

$$\left[(d+h-x)^3\right]_d^{d+h} = (d+h-d-h)^3 - (d+h-d)^3 = -h^3$$

(a) $W_2 = \frac{\epsilon_0 l w h^3}{3}\left(\frac{\mu_0 k}{2}\right)^2$ $\qquad W_1 = 0$

(b) Poynting vector $= \frac{\text{Power}}{\text{area}} = S = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$



$$S = \frac{1}{\mu_0} E \cdot \frac{E}{c} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 k}{2}\right)^2 (ct - |x|)^2$$

$S = \frac{\mu_0 k^2}{4c}(ct - |x|)^2 \,\hat{x}$. Power enters the bottom of box at $x = d$:

$$\frac{\text{Energy}}{\text{time}} = \text{Power} = \int S_x \cdot dA_x = S \cdot l w = \frac{\mu_0 k^2 l w}{4c}(ct - d)^2$$

How much power leaves the top of the box?

$$S(x = d+h, t = 2) = \frac{\mu_0 k^2}{4c}\left(c\,\frac{d+h}{c} - (d+h)\right)^2 = 0 \quad - \text{NONE}$$

(c) Energy flowing into box $= \int_{t_1}^{t_2} \text{Power} \cdot dt$

$$= \frac{\mu_0 k^2 l w}{4c}\int_{d/c}^{d+h/c} (ct-d)^2\, dt = \frac{\mu_0 k^2 l w}{4c}\, \frac{(ct-d)^3}{3c}\Bigg|_{d/c}^{\frac{d+h}{c}}$$

$$(d+h-d)^3 - (d-d)^3 = h^3$$

$$W = \frac{\mu_0 k^2 l w h^3}{12 c^2}\left(\frac{\mu_0 \epsilon_0}{1/c^2}\right) = \mu_0^2 \epsilon_0 k^2 l w h^3 / 12 \quad \text{same as (b)} \checkmark$$

$\sigma\tau.$

Problem 10.3 Find the fields, and the charge and current distributions, corresponding to

$$V(\mathbf{r}, t) = 0, \quad \mathbf{A}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}}.$$ Spherical

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = 0 + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}, \quad \mathbf{B} = \nabla\times\bar{A} = 0$$

$$(\nabla\times A)_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r} = 0$$

Similarly $(\nabla\times A)_\theta = 0$ since $\frac{\partial A_r}{\partial\phi} = 0$, $(\nabla\times A)_\phi = 0$ since $\frac{\partial A_r}{\partial\theta} = 0$

This is just a stationary point charge at the origin!

Problem 10.5 Use the gauge function $\lambda = -(1/4\pi\epsilon_0)(qt/r)$ to transform the potentials in Prob. 10.3, and comment on the result.

$$A' = A + \nabla\lambda, \qquad V' = V - \frac{\partial\lambda}{\partial t}$$

$$\frac{\partial\lambda}{\partial t} = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}$$

$$\nabla\lambda = \frac{\partial\lambda}{\partial r}\hat{r} = +\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}$$

sign? $$A' = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} + \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} = 0 \checkmark$$

$$V' = 0 - \frac{-1}{4\pi\epsilon_0}\frac{q}{r} = \frac{q}{4\pi\epsilon_0 r} \checkmark$$

$$\mathbf{E} = -\nabla V - \frac{\partial A}{\partial t} = -\frac{q\hat{r}}{4\pi\epsilon_0}\frac{\partial}{\partial r}r^{-1} - 0 = \frac{q\hat{r}}{4\pi\epsilon_0 r^2} \checkmark$$

$$\mathbf{B} = \nabla\times A = 0 \checkmark$$

These are the usual fields and potentials for a point charge.

Problem 10.6 Which of the potentials in Ex. 10.1, Prob. 10.3, and Prob. 10.4 are in the Coulomb gauge? Which are in the Lorentz gauge? (Notice that these gauges are not mutually exclusive.)

p. 421 Coulomb gauge: $\nabla \cdot A = 0$ $\qquad A' = A + \nabla \lambda$

Lorentz gauge: $\nabla \cdot A = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$ $\qquad V' = V - \frac{\partial \lambda}{\partial t}$

Example 10.1

Find the charge and current distributions that would give rise to the potentials

$$V = 0, \quad \mathbf{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2\,\hat{\mathbf{z}}, & \text{for } |x| < ct, \\ 0, & \text{for } |x| > ct, \end{cases}$$ Cartesian coords

where k is a constant, and $c = 1/\sqrt{\epsilon_0 \mu_0}$.

$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$: COULOMB

$\frac{\partial V}{\partial t} = 0 = \nabla \cdot A$: ALSO LORENTZ

Problem 10.3 Find the fields, and the charge and current distributions, corresponding to

$$V(\mathbf{r}, t) = 0, \quad \mathbf{A}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}}.$$ Spherical

$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{r^2}\right) = 0$; COULOMB

$\frac{\partial V}{\partial t} = 0 = \nabla \cdot A$; ALSO LORENTZ

OR: $\nabla \cdot A \sim \nabla \cdot \left(\frac{\hat{r}}{r^2}\right)$
$\sim \delta^3(\mathbf{r})$
NEITHER?

Problem 10.4 Suppose $V = 0$ and $\mathbf{A} = A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}}$, where A_0, ω, and k are constants.

Cartesian.

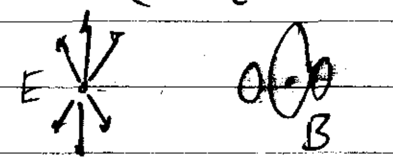
$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$; COULOMB

$\frac{\partial V}{\partial t} = 0 = \nabla \cdot A$: ALSO LORENTZ

Problem 10.25 A particle of charge q is traveling at constant speed v along the x axis. Calculate the total power passing through the plane $x = a$, at the moment the particle itself is at the origin. [*Answer:* $q^2 v/32\pi\epsilon_0 a^2$]

(10.68) 439 From Example 10.4, we have $\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1-\frac{v^2}{c^2}}{\left(1-\frac{v^2\sin^2\theta}{c^2}\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}$.

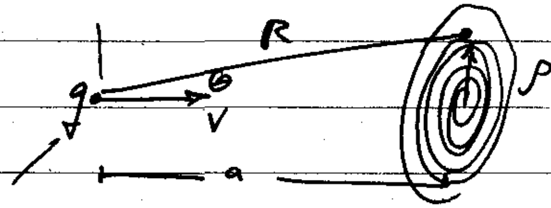
(10.69) 440 and $\mathbf{B} = \frac{1}{c}(\hat{r}\times\mathbf{E}) = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E})$



Figs 10.10 & 11

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$ is the vector from the present location of the particle to $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$

Fig 10.9 434



We will integrate over the whole plane at $x = a$

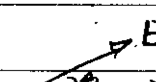
$\int d\mathbf{A} = \int_0^\infty 2\pi\rho\, d\rho\ \hat{x}$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}\mathbf{E}\times\left(\frac{1}{c^2}(\mathbf{v}\times\mathbf{E})\right) = \frac{1}{\mu_0 c^2}\left[\mathbf{E}\times(\mathbf{v}\times\mathbf{E})\right]$$

$c^2 = \frac{1}{\mu_0\epsilon_0}$ and $A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$ so

$$\mathbf{S} = \epsilon_0\left[\mathbf{v}(\mathbf{E}\cdot\mathbf{E}) - \mathbf{E}(\mathbf{v}\cdot\mathbf{E})\right] = \epsilon_0\left[E^2\mathbf{v} - \mathbf{E}(\mathbf{v}\cdot\mathbf{E})\right] = \frac{\text{power}}{\text{area}}$$

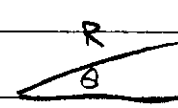
$\text{Power} = \int S_x \cdot dA_x$


$E_x = E\cos\theta$

$$S_x = \epsilon_0\left[E^2 v - E_x(vE_x)\right] = \epsilon_0 v\left[E^2 - E_x^2\right] = \epsilon_0 v\left[E^2 - E^2\cos^2\theta\right]$$

$$S_x = \epsilon_0 v\left[E^2\sin^2\theta\right]$$

$$E^2 = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{\left(1-\frac{v^2}{c^2}\right)^2}{\left(1-\frac{v^2\sin^2\theta}{c^2}\right)^3}\frac{1}{R^4}$$



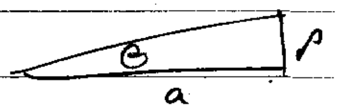
$a = R\cos\theta$

$\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}$ $\quad E^2 = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{\gamma^4}\left(\frac{\cos\theta}{a}\right)^4 \frac{1}{\left(1-\frac{v^2\sin^2\theta}{c^2}\right)^3}$ $\quad \frac{1}{R} = \frac{\cos\theta}{a}$

$$S_n = \epsilon_0 v E^2 \sin^2\theta = \epsilon_0 v \sin^2\theta \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{\gamma^4}\left(\frac{\cos\theta}{a}\right)^4 \frac{1}{\left(1-\frac{v^2\sin^2\theta}{c^2}\right)^3}$$

$$S_n = \frac{\epsilon_0 v}{\gamma^4}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \sin^2\theta \left(\frac{\cos\theta}{a}\right)^4 \left[\frac{1}{\left(1-\frac{v^2\sin^2\theta}{c^2}\right)^3}\right]$$

$dA = 2\pi\rho\, d\rho$



$\frac{\rho}{a} = \tan\theta$

$d\rho = \frac{a}{\cos^2\theta}\, d\theta$

$$dA = 2\pi\left(a\frac{\sin\theta}{\cos\theta}\right)\frac{a}{\cos^2\theta}\, d\theta$$

limits: (1) $\int_0^\infty d\rho \rightarrow \int_0^{\pi/2} d\theta$

$$dA = 2\pi\, a^2 \frac{\sin\theta}{\cos^3\theta}\, d\theta$$

$$S_n\, dA = \left\{\frac{\epsilon_0 v}{\gamma^4}\left(\frac{q}{4\pi\epsilon_0}\right)^2\right\} \sin^2\theta\left(\frac{\cos\theta}{a}\right)^4 2\pi a^2 \frac{\sin\theta}{\cos^3\theta}\left[\;\right] d\theta$$

$$= \left\{\quad\right\} 2\pi\, \frac{\sin^3\theta}{a^2}\cos\theta \left[\;\right] d\theta$$

$$2\pi\left\{\quad\right\} = \frac{2\pi\epsilon_0 v}{\gamma^4}\frac{q^2}{4\cdot 4\cdot\pi^2\epsilon_0^2} = \frac{v q^2}{\gamma^4\, 8\pi\epsilon_0} = K$$

$$\int S_n\, dA = \frac{K}{a^2}\int_0^{\pi/2} \sin^3\theta\cos\theta\left[1-\frac{v^2\sin^2\theta}{c^2}\right]^{-3} d\theta = \frac{K}{a^2}\int I\, d\theta$$

Let $u = \sin^2\theta$, $du = 2\sin\theta\, d\theta\cos\theta$, $g = \frac{v^2}{c^2}$

$$\int I\, d\theta = \int \tfrac{1}{2}\, u\, du\,[1-gu]^{-3} = \frac{1}{2}\left(\frac{1}{-g}\right)^2\left[\frac{-1}{1-gu} + \frac{1}{2(1-gu)^2}\right]$$

$a = 1,\ b = -g$

Dwight $\frac{91.3}{23}$ $\int \frac{x\, dx}{X^3} = \frac{1}{b^2}\left[-\frac{1}{X} + \frac{a}{2X^2}\right]$ where $X = a + bx$

limits: $\int_{\theta=0}^{\pi/2} d\theta = \int_{u=0}^{1} du \qquad u = \sin^2\theta, \; u(\tfrac{\pi}{2}) = \sin^2(\tfrac{\pi}{2}) = 1$

$$\int_0^{\pi/2} I\,d\theta = \frac{1}{2g^2}\left[\frac{-2(1-gu)}{2(1-gu)^2} + \frac{1}{2(1-gu)^2}\right]_0^{u=1} = \frac{1}{4g^2}\left[\frac{-1+2gu}{(1-gu)^2}\right]_0^{u=1}$$

$$= \frac{1}{4g^2}\left[\left(\frac{-1+2g}{(1-g)^2}\right) - \left(\frac{-1+0}{1^2}\right)\right] = \frac{1}{4g^2}\left[\frac{2g-1}{(1-g)^2} + 1\right].$$

$$\left[\frac{2g-1}{(1-g)^2} + 1\right] = \frac{2g-1}{(1-g)^2} + \frac{1^2+g^2-2g}{(1-g)^2} = \left[\frac{g^2}{(1-g)^2}\right]$$

$$\int I\,d\theta = \frac{1}{4g^2}\left[\frac{g^2}{(1-g)^2}\right] = \frac{1}{4(1-g)^2} = \frac{1}{4(1-\frac{v^2}{c^2})^2} = \frac{\gamma^4}{4}$$

$$\text{Power} = \int S_r\,dA = \frac{k}{a^2}\int I\,d\theta \quad \text{where } k = \frac{vq^2}{8\pi\epsilon_0\gamma^4}$$

$$\text{Power} = \frac{vq^2}{8\pi\epsilon_0\gamma^4 a^2}\left(\frac{\gamma^4}{4}\right) = \frac{vq^2}{32\pi\epsilon_0 a^2}$$