EM 2a

Physical Systems – spring 2007

EM HW 2a Tues. 10 April 2007 Griffiths **Ch.7 #34, 42, 53.** Extra: 58 (a-c)

Problem 7.34 Suppose

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\,\hat{\mathbf{r}};\quad \mathbf{B}(\mathbf{r},t)=0$$

(the theta function is defined in Prob. 1.45b). Show that these fields satisfy all of Maxwell's equations, and determine ρ and $\mathbf{J}$. Describe the physical situation that gives rise to these fields.

(b) Let $\theta(x)$ be the **step function**:

$$\theta(x) \equiv \left\{\begin{array}{ll} 1, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{array}\right\}. \tag{1.95}$$

Show that $d\theta/dx = \delta(x)$.

Problem 7.42 In a **perfect conductor**, the conductivity is infinite, so $\mathbf{E} = 0$ (Eq. 7.3), and any net charge resides on the surface (just as it does for an *im*perfect conductor, in electro*statics*).

(a) Show that the magnetic field is constant ($\partial\mathbf{B}/\partial t = 0$), inside a perfect conductor.

(b) Show that the magnetic flux through a perfectly conducting loop is constant.

A **superconductor** is a perfect conductor with the *additional* property that the (constant) $\mathbf{B}$ inside is in fact *zero*. (This "flux exclusion" is known as the **Meissner effect**.[18])

(c) Show that the current in a superconductor is confined to the surface.

(d) Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{\mathbf{z}}$ while cooling it below T_c. Find the induced surface current density $\mathbf{K}$, as a function of the polar angle θ.

Problem 7.53 Two coils are wrapped around a cylindrical form in such a way that the *same flux passes through every turn of both coils.* (In practice this is achieved by inserting an iron core through the cylinder; this has the effect of concentrating the flux.) The "primary" coil has N_1 turns and the secondary has N_2 (Fig. 7.54). If the current I in the primary is changing, show that the emf in the secondary is given by

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}, \tag{7.67}$$

where $\mathcal{E}_1$ is the (back) emf of the primary. [This is a primitive **transformer**—a device for raising or lowering the emf of an alternating current source. By choosing the appropriate number of turns, any desired secondary emf can be obtained. If you think this violates the conservation of energy, check out Prob. 7.54.]

7.34. $\mathbf{E}(\mathbf{r},t) = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\,\hat{r}$, $\mathbf{B}(\mathbf{r},t)=0$

where Heaviside fxn $\theta(x) \equiv \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x\le 0 \end{cases}$, $\frac{d\theta}{dx} = \delta(x)$

ⓐ Show that these fields satisfy all of Maxwell's equations

$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$ $\quad \nabla\cdot\mathbf{B} = 0$ $\quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ $\quad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$

ⓑ Determine charge density ρ and current density $\mathbf{J}$.

ⓒ Describe the physical situation that gives rise to these fields.

$$\nabla\times\mathbf{B} = 0 = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\frac{d\mathbf{E}}{dt} = \frac{d}{dt}\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\,\hat{r}\right) = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}\,\frac{d\theta}{dt}$$

$\frac{d\theta}{dx} = \delta(x)$ so $\frac{d}{dt}\theta(vt-r) = \delta(vt-r)\,v$

$$\frac{d\mathbf{E}}{dt} = -\frac{1}{4\pi\epsilon_0}\frac{vq}{r^2}\delta(vt-r)\,\hat{r}$$

$$\mu_0\mathbf{J} = -\mu_0\epsilon_0\frac{d\mathbf{E}}{dt}$$

$$\mathbf{J} = -\epsilon_0\left(-\frac{1}{4\pi\epsilon_0}\right)\frac{q}{r^2}\delta(vt-r)\,\hat{r}$$

$$\mathbf{J} = \frac{qv}{4\pi r^2}\delta(vt-r)\,\hat{r}$$

Radially outward current for $vt>r$; no current for $vt \le r$

$\mathbf{E}$ looks like the field of a negative point charge, except the $\theta(vt-r)$ term has it expanding out to a spherical shell of radius $r=vt$. Outside $r=vt$, $E=0$, so the shell must have $+q$.

$$\nabla\cdot E = \frac{\rho}{\epsilon_0} = \frac{-q}{4\pi\epsilon_0}\nabla\cdot\frac{\theta(vt-r)\,\hat{r}}{r^2}$$

$$\nabla\cdot\frac{\theta(vt-r)\,\hat{r}}{r^2} = \theta(vt-r)\,\nabla\cdot\frac{\hat{r}}{r^2} + \frac{1}{r^2}\hat{r}\cdot\nabla\theta(vt-r)$$

$$= \theta(vt-r)\left(4\pi\delta^3(r)\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\theta(vt-r)$$

1.99 p. 50 prob 1.45

$$= 4\pi\,\delta^3(r)\,\theta(vt-r) + \frac{1}{r^2}\left(-\delta(vt-r)\right)$$

$$= 4\pi\,\delta^3(r)\,\theta(vt) + -\frac{1}{r^2}\,\delta(vt-r)$$

(r = 0)

$$\nabla\cdot E = \frac{\rho}{\epsilon_0} = \frac{-q}{4\pi\epsilon_0}\left[4\pi\,\delta^3(r)\,\theta(vt) - \frac{1}{r^2}\delta(vt-r)\right]$$

$$\rho = -q\left[\delta^3(r)\,\theta(vt) - \frac{1}{4\pi r^2}\delta(vt-r)\right]$$

$$\rho = -q\delta^3(r)\,\theta(vt) + \frac{q}{4\pi r^2}\delta(vt-r)$$

For $t<0$, the $\rho=0$ everywhere (and $E=0$)

$\nabla\times E = \nabla\times E\hat{r} = 0$; purely radial E cannot curl

$B=0$ so $\nabla\cdot B = 0$

7.36 Suppose a magnetic monopole q_m passes through
328 a resistanceless loop of wire with self-inductance
L. What current is induced in the loop?

[gen]eralized: $\bar{\nabla}\times\bar{E} = -\mu_0 J_m - \frac{\partial \bar{B}}{\partial t}$ Integrate over the surface of the loop:

$$\int(\bar{\nabla}\times\bar{E})\cdot d\bar{a} = \oint \bar{E}\cdot d\bar{l} = \mathcal{E} = -\mu_0\int J_m \cdot da - \frac{d}{dt}\int B\cdot da$$
$$= -\mu_0 I_{m_{enclosed}} - \frac{d\Phi}{dt}$$

$$\mathcal{E} = -L\frac{dI}{dt} \quad \text{so} \quad \frac{dI}{dt} = \frac{\mu_0}{L} I_{m_{enc}} + \frac{1}{L}\frac{d\Phi}{dt}$$

Let $I_{m_{enclosed}} = \frac{\Delta Q_m}{\Delta t}$ and $\frac{d\Phi}{dt} = \frac{\Delta\Phi}{\Delta t}$ where

$\Delta\Phi$ = change in flux through loop.

If we use the flat surface of the loop, then $\Delta Q_m = q_m$
and $\Delta\Phi = 0$ ($\Phi = 0$ when monopole is far away

$\Phi = \mu_0 q_m/2$ just before it passes through loop

$\Phi = -\mu_0 q_m/2$ just after passing through

$\Phi \to 0$ as it moves far away again

Solve for $I(q_m, L) = \frac{\mu_0}{L}\Delta Q_m + \frac{\Delta\Phi}{L}$ ($\Delta\Phi \to 0$)

$$I = \frac{\mu_0 q_m}{L}$$

Problem 7.42 In a **perfect conductor**, the conductivity is infinite, so $\mathbf{E} = 0$ (Eq. 7.3), and any net charge resides on the surface (just as it does for an *imperfect* conductor, in electro*statics*).

(a) Show that the magnetic field is constant ($\partial \mathbf{B}/\partial t = 0$), inside a perfect conductor.

(b) Show that the magnetic flux through a perfectly conducting loop is constant.

(a) Faraday's law: $\frac{\partial \bar{B}}{\partial t} = -\bar{\nabla} \times \bar{E}$: if $\bar{E} = 0$ then $\bar{\nabla} \times \bar{E} = 0$ so

$\frac{\partial B}{\partial t} = 0 \rightarrow B =$ constant inside perfect conductor.

(b) Faraday's law: $\oint \bar{E} \cdot d\bar{l} = -\frac{d\phi}{dt} = 0$ if $\bar{E} = 0$, so

$\Phi =$ constant through perfectly conducting loop.

A **superconductor** is a perfect conductor with the *additional* property that the (constant) **B** inside is in fact *zero*. (This "flux exclusion" is known as the Meissner effect.[18])

(c) Show that the current in a superconductor is confined to the surface.

Ampere's law: $\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enclosed}$. If there were current inside the conductor, then you could draw an amperian loop INSIDE which enclosed nonzero I, and B_{in} would be nonzero.

Since $B_{in} = 0 \rightarrow I_{in} = 0$.

[Even if you use $\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$, $\bar{E} = 0 = \bar{B}$ so $\bar{J}_{inside} = 0$]

NO CURRENT INSIDE $\rightarrow$ ANY CURRENT MUST BE ON SURFACE

(d) Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{z}$ while cooling it below T_c. Find the induced surface current density **K**, as a function of the polar angle θ.

Problem 7.59 In a **perfect conductor**, the conductivity is infinite, so $E = 0$ (equation 7.3), and any net charge resides on the surface (just as it does for an *im*perfect conductor, in electro*statics*).

(a) Show that the magnetic field is constant (i.e., $\partial \mathbf{B}/\partial t = 0$), inside a perfect conductor.

FARADAY'S LAW: $\frac{\partial \bar{B}}{\partial t} = -\nabla \times \bar{E}$ if $\bar{E} = 0$ then $\nabla \times \bar{E}$

So $\bar{B}$ = constant inside a conductor.

A **superconductor** is a perfect conductor with the *additional* property that this constant **B** is always *zero*. (This "flux exclusion" is known as the **Meissner effect.**)

(b) Show that the current in a superconductor is confined to the surface.

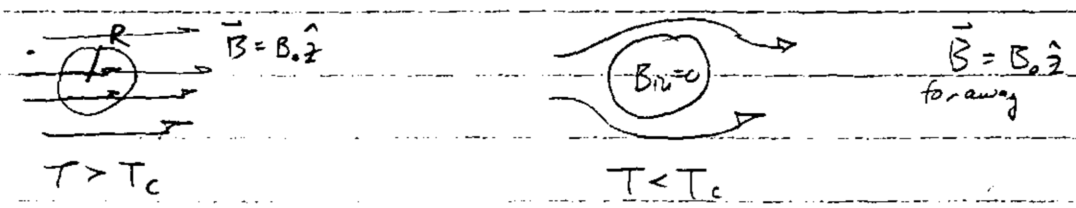
AMPERE'S LAW: $\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enclosed}$.

If there were current inside the conductor, then you could draw an amperian loop INSIDE which enclosed nonzero I, and B_{in} would be nonzero.

Since $B_{in} = 0$, therefore $I_{in} = 0$

Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another.

(c) Suppose you had a sphere (radius R) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{z}$ while cooling it below T_c. Find the induced surface current density **K**, as a function of the polar angle θ.



Flux is expelled as the material cools below T_c.

To find $\bar{K}$, apply BC: $H_{tan} \rightarrow \frac{B_{out\parallel}}{\mu_{out}} - \frac{B_{in\parallel}}{\mu_{in}} = K$

$B_{perp} \rightarrow B_{\perp out} = B_{\perp in}$

Far away ($r\to\infty$) $\bar{B} = B_o \hat{k} = -\mu_o \nabla \varphi^* = -\mu_o \left[\frac{\partial \varphi^*}{\partial z}\hat{k}\right]$

so the dominant term in the φ^* series is

$\varphi^*_1 = -\frac{B_o}{\mu_o} z = -\frac{B_o}{\mu_o} r\cos\theta$

Since B is finite at $r\to\infty$, $\varphi(r\to\infty)$ can't blow up ($A_{l>1}=0$)

$\varphi^*_{out} = \varphi^*_1 + \sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} P_l = \frac{-B_o}{\mu_o} r\cos\theta + \sum \frac{C_l}{r^{l+1}} P_l$

Substitute this into $\underline{\vec{B} = -\mu_o \nabla \varphi^*}$ to find form of $\bar{B}$:

$\bar{B}_{out} = -\mu_o \bar{\nabla}\left[\frac{-B_o}{\mu_o} r\cos\theta + \sum \frac{C_l}{r^{l+1}} P_l\right]$

$B_{r_{out}} = -\mu_o \frac{\partial}{\partial r}\left[\frac{-B_o}{\mu_o} r\cos\theta + \sum \frac{C_l}{r^{l+1}} P_l\right]$

$= B_o \cos\theta + \mu_o \sum \frac{(l+1) C_l P_l}{r^{l+2}}$

Apply this to boundary at $r=a$:

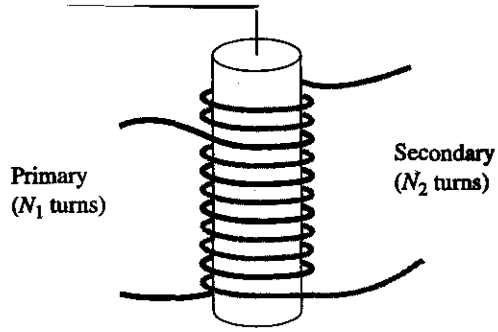
$B_r(a) = B_o\cos\theta + \mu_o \sum \frac{(l+1) C_l P_l}{a^{l+2}}$

Since $B_\perp = B_\perp$ = continuous across boundary, the only term here is $P_1 = \cos\theta$ – all other l's vanish:

Problem 7.53 Two coils are wrapped around a cylindrical form in such a way that the *same flux passes through every turn of both coils.* (In practice this is achieved by inserting an iron core through the cylinder; this has the effect of concentrating the flux.) The "primary" coil has N_1 turns and the secondary has N_2 (Fig. 7.54). If the current I in the primary is changing, show that the emf in the secondary is given by

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}, \tag{7.67}$$

where $\mathcal{E}_1$ is the (back) emf of the primary. [This is a primitive **transformer**—a device for raising or lowering the emf of an alternating current source. By choosing the appropriate number of turns, any desired secondary emf can be obtained. If you think this violates the conservation of energy, check out Prob. 7.54.]



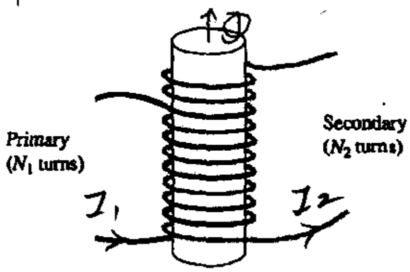
Flux through one loop of primary = Flux through one loop of secondary

$$\Phi_1/N_1 = \Phi = \Phi = \Phi_2/N_2$$

$$\mathcal{E}_1 = -N_1 \frac{d\Phi}{dt} \quad \text{and} \quad \mathcal{E}_2 = -N_2 \frac{d\Phi}{dt}$$

$$\frac{d\Phi}{dt} = \frac{d\Phi}{dt}$$

$$\frac{\mathcal{E}_1}{N_1} = \frac{\mathcal{E}_2}{N_2} \longrightarrow \frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{N_1}{N_2} \checkmark$$



Let Φ = flux through each turn

Φ_1 = total flux in primary $= N_1 \Phi$

Φ_2 = total flux in secondary $= N_2 \Phi$

Problem 7.54 A transformer (Prob. 7.53) takes an input AC voltage of amplitude V_1, and delivers an output voltage of amplitude V_2, which is determined by the turns ratio ($V_2/V_1 = N_2/N_1$). If $N_2 > N_1$ the output voltage is greater than the input voltage. Why doesn't this violate conservation of energy? *Answer:* Power is the product of voltage and current; evidently if the voltage goes *up*, the current must come *down*. The purpose of this problem is to see exactly how this works out, in a simplified model.

(a) In an ideal transformer the same flux passes through all turns of the primary and of the secondary. Show that in this case $M^2 = L_1 L_2$, where M is the mutual inductance of the coils, and L_1, L_2 are their individual self-inductances.

(b) Suppose the primary is driven with AC voltage $V_{in} = V_1 \cos(\omega t)$, and the secondary is connected to a resistor, R. Show that the two currents satisfy the relations

$$L_1 \frac{dI_1}{dt} + M\frac{dI_2}{dt} = V_1 \cos(\omega t); \quad L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt} = -I_2 R.$$

(c) Using the result in (a), solve these equations for $I_1(t)$ and $I_2(t)$. Assume I_1 has no DC component. $I_2 = -\frac{V_1 L_2}{MR}\cos\omega t$ $\quad I_1 = \frac{V_1}{\omega L_1}\sin\omega t + \frac{V_1 L_2}{L_1 R}\cos\omega t$

(d) Show that the output voltage ($V_{out} = I_2 R$) divided by the input voltage (V_{in}) is equal to the turns ratio: $V_{in}/V_{out} = N_2/N_1$.

$\langle \cos\omega t \cdot \sin\omega t \rangle = 0$

(e) Calculate the input power ($P_{in} = V_{in} I_1$) and the output power ($P_{out} = V_{out} I_2$) and show that their averages over a full cycle are equal. Hint $\langle \cos^2\omega t\rangle = \langle \sin^2\omega t \rangle = \frac{1}{2}$

(a) $\Phi_1 = N_1 \Phi = I_1 L_1 + M I_2 \rightarrow \Phi = I_1 \frac{L_1}{N_1} + I_2 \frac{M}{N_1}$

$\Phi_2 = N_2 \Phi = I_2 L_2 + M I_1 \rightarrow \Phi = I_2 \frac{L_2}{N_2} + I_1 \frac{M}{N_2}$

subtract $\quad -\left(\Phi = I_2 \frac{M}{N_1} + I_1 \frac{L_1}{N_1}\right)$

$I_2\left(\frac{L_2}{N_2} - \frac{M}{N_1}\right) = I_1\left(\frac{M}{N_2} - \frac{L_1}{N_1}\right)$

If $I_2 = 0$ then $\frac{M}{N_2} = \frac{L_1}{N_1} \rightarrow \frac{M}{L_1} = \frac{N_2}{N_1}$

If $I_1 = 0$ then $\frac{M}{N_1} = \frac{L_2}{N_2} \rightarrow \frac{L_2}{M} = \frac{N_2}{N_1}$ ✪

subtract $\frac{M}{L_1} - \frac{L_2}{M} = 0 \rightarrow \frac{M}{L_1} = \frac{L_2}{M} \rightarrow M^2 = L_1 L_2$

(b) If primary is driven with $V_{in} = V_1 \cos\omega t$, then

back emf mutual inductance

(1) $V_1 \cos\omega t - L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt} = 0$

If secondary is connected to a resistor then

induction back emf voltage drop

(2) $-M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt} + I_2 R = 0$ $\quad L_2 \frac{dI_2}{dt} = -I_2 R - M \frac{dI_1}{dt}$

(c) Solve for I_1 and I_2

$L_2 \times (1)$: $V_1 L_2 \cos\omega t - L_1 L_2 \frac{dI_1}{dt} - M L_2 \frac{dI_2}{dt} = 0$

$V_1 L_2 \cos\omega t - L_1 L_2 \frac{dI_1}{dt} + M \left(M \frac{dI_1}{dt} + I_2 R \right) = 0$

$L_1 L_2 = M^2$

$V_1 L_2 \cos\omega t + M I_2 R = 0$

$I_2 = -\frac{V_1 L_2 \cos\omega t}{MR}$

$\frac{dI_2}{dt} = +\frac{V_1 L_2 \omega}{MR} \sin\omega t$

Plug these into either (1) or (2) to get

(1) $\frac{dI_1}{dt} = \frac{V_1}{L_1} \cos\omega t - \frac{M}{L_1} \frac{dI_2}{dt} = \frac{V_1}{L_1} \cos\omega t - \frac{M}{L_1} \frac{V_1 L_2 \omega}{MR} \sin\omega t$

$I_1 = \int \left(\frac{V_1}{L_1} \cos\omega t - \frac{V_1 L_2 \omega}{L_1 R} \sin\omega t \right) dt$

$= \frac{V_1}{\omega L_1} \sin\omega t + \frac{V_1 L_2}{L_1 R} \cos\omega t + C$

$C = 0$ if there is no DC component

→

EXTRA:

Problem 7.58 A certain transmission line is constructed from two thin metal "ribbons," of width w, a very small distance $h \ll w$ apart. The current travels down one strip and back along the other. In each case it spreads out uniformly over the surface of the ribbon.

(a) Find the capacitance per unit length, $\mathcal{C}$.

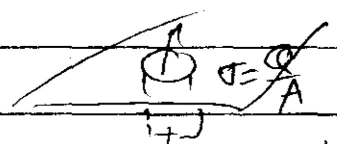
(b) Find the inductance per unit length, $\mathcal{L}$.

$$\mathcal{L}\mathcal{C} = \frac{\mu_0 h}{w} \cdot \frac{w\epsilon_0}{h} = \mu_0\epsilon_0$$

(c) What is the product $\mathcal{L}\mathcal{C}$, numerically? [$\mathcal{L}$ and $\mathcal{C}$ will, of course, vary from one kind of transmission line to another, but their *product* is a universal constant—check, for example, the cable in Ex. 7.13—provided the space between the conductors is a vacuum. In the theory of transmission lines, this product is related to the speed with which a pulse propagates down the line: $v = 1/\sqrt{\mathcal{L}\mathcal{C}}$.]

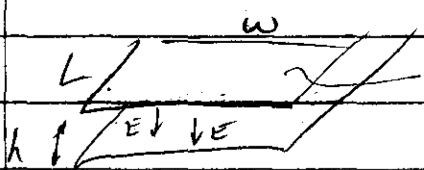
$V = \sqrt{\frac{1}{\mu_0\epsilon_0}} = c$ = speed of light

(a)



$\sigma = \frac{Q}{A}$

$\oint E\cdot dA = \frac{Q}{\epsilon_0}$ $\quad 2E A = \frac{\sigma A}{\epsilon_0} \rightarrow E = \frac{\sigma}{2\epsilon_0}$

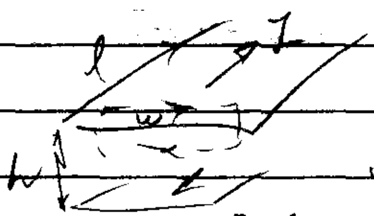


$A' = wL, \quad Q = \sigma A' = \sigma w l$

$E_{tot} = \frac{\sigma}{\epsilon_0} = \frac{V}{h}$ $\quad C = \frac{Q}{V} = \frac{\sigma w l}{h\sigma/\epsilon_0} = \frac{w\epsilon_0 l}{h}$

$\frac{C}{l} = \frac{w\epsilon_0}{h} = \mathcal{C}$

(b)



$\oint B\cdot dl = \mu_0 I = 2Bw \rightarrow B_{each} = \frac{\mu_0 I}{2w}$

$B_{tot} = 2B_{each} = \frac{\mu_0 I}{w}$

vol = wlh $\quad W = \frac{1}{2}LI^2$ $\quad$ and $\quad \frac{W}{vol} = \frac{B^2}{2\mu_0}$

stored magnetic energy

$\frac{W}{wlh} = \left(\frac{\mu_0 I}{w}\right)^2 \frac{1}{2\mu_0}$

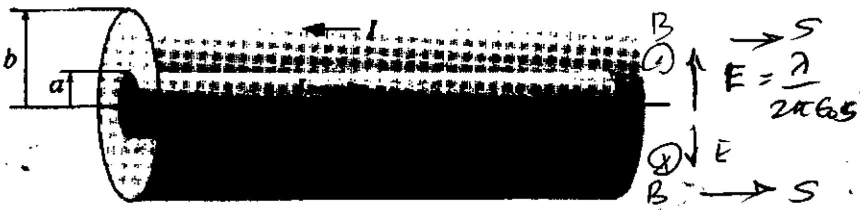
$\frac{1}{2}LI^2 = \frac{\mu_0^2 I^2}{w^2 2\mu_0} wlh \rightarrow L = \frac{\mu_0 l h}{w} \rightarrow \frac{L}{l} = \frac{\mu_0 h}{w} = \mathcal{L}$

EM HW 2b Thus.12 April 2007 Griffiths **Ch.8** # 1(a). Extra: #1(b)

Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.58, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).

Example 7.13

A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b) as shown in Fig. 7.39. Find the magnetic energy stored in a section of length l.



Solution: According to Ampère's law, the field between the cylinders is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}.$$

Elsewhere, the field is zero. Thus, the energy per unit volume is

$$\frac{dW}{d\tau} = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}.$$

The energy in a cylindrical shell of length l, radius s, and thickness ds, then, is $d\tau = 2\pi l\, s\, ds$

$$dW = \left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right) 2\pi l s\, ds = \frac{\mu_0 I^2 l}{4\pi}\left(\frac{ds}{s}\right).$$

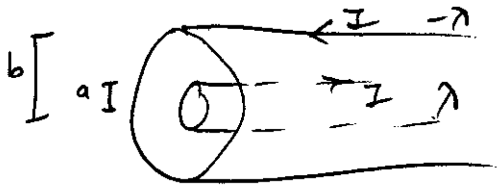
Integrating from a to b, we have:

$$W = \frac{\mu_0 I^2 l}{4\pi}\ln\left(\frac{b}{a}\right).$$

By the way, this suggests a very simple way to calculate the self-inductance of the cable. According to Eq. 7.29, the energy can also be written as $\frac{1}{2}LI^2$. Comparing the two expressions,[12]

$$L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right).$$

This method of calculating self-inductance is especially useful when the current is not confined to a single path, but spreads over some surface or volume. In such cases different parts of the current may circle different amounts of flux, and it can be very tricky to get L directly from Eq. 7.25.



8.1 (a) $\frac{\text{Power}}{\text{Area}} = S = \frac{|\mathbf{E}\times\mathbf{B}|}{\mu_0}$ So Power $= \int S \cdot da$

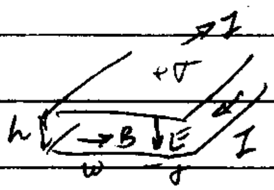
$$P = \int_a^b S \cdot 2\pi s \, ds$$

$$S = \frac{1}{\mu_0} \frac{\lambda}{2\pi\epsilon_0 s} \hat{s} \times \frac{\mu_0 I}{2\pi s} \hat{\phi} = \frac{\lambda I}{4\pi^2 \epsilon_0 s^2} \hat{z}$$

$$P_1 = \int_a^b S \cdot 2\pi s \, ds = \frac{2\pi}{4\pi^2\epsilon_0} \lambda I \int_a^b \frac{s}{s^2} ds = \frac{\lambda I}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

$$V = \int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0} \int_a^b \frac{ds}{s} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

So $P_2 = IV = \frac{\lambda I}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$ ✓ $= P_1$

(b) 

We found $\mathbf{E} = \frac{\sigma}{\epsilon_0} = \frac{V}{h}$ in direction drawn

and $B = \frac{\mu_0 I}{w}$ in direction drawn

$$\text{Power} = \int S \cdot da = S w h \quad \text{where } S = \frac{1}{\mu_0} |\mathbf{E}\times\mathbf{B}| = \frac{\sigma I}{\epsilon_0 w}$$

$$P_1 = \frac{\sigma I}{\epsilon_0 w} wh = \frac{\sigma I h}{\epsilon_0}$$

$$V = \int \mathbf{E} \cdot d\mathbf{l} = \frac{\sigma h}{\epsilon_0}$$

So $P_2 = IV = \frac{I\sigma h}{\epsilon_0} = P_1$

Griffiths p.349

Problem 8.2 Consider the charging capacitor in Prob. 7.31.

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector **S** in the gap. Note especially the *direction* of **S**. Check that Eq. 8.14 is satisfied.

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9—in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

Problem 7.31 A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in Fig. 7.43. Find the magnetic field in the gap, at a distance $s < a$ from the axis.

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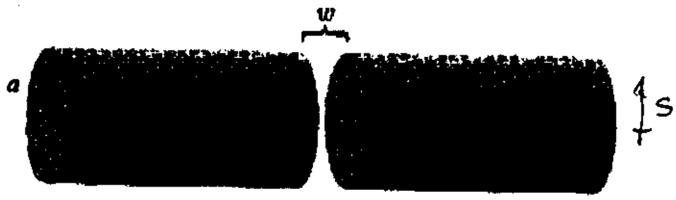


The magnetic field in the gap, due to displacement current, is CONTINUOUS with the magnetic field in the wire, due to flowing charge

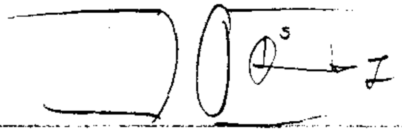
$\oint B \cdot dl = \mu_0 I_{enclosed}$ and $\frac{I_{enc}}{I_{tot}} = \frac{\pi s^2}{\pi a^2} \rightarrow I_{enc} = \frac{I s^2}{a^2}$

$B \cdot 2\pi s = \mu_0 I \frac{s^2}{a^2}$

$B(s) = \mu_0 I s / 2\pi a^2$

$\text{Power}_1 = IV$ and $V = E_1 w = I t w / \epsilon_0 \pi a^2$ (next page)

So $\text{Power}_1 = \frac{I^2 t w}{\epsilon_0 \pi a^2}$



$$\oint B \cdot dl = \mu_0 I_{enc}$$

$$B \cdot 2\pi s = \mu_0 I \frac{s^2}{a^2}$$

$$\frac{I_{enc}}{I} = \frac{\pi s^2}{\pi a^2} \rightarrow I_{enc} = \frac{I s^2}{a^2}$$

$$B(s) = \frac{\mu_0 I s^2}{2\pi s a^2} = \frac{\mu_0 I s}{2\pi a^2}$$

Now find $E(t)$: $E = \frac{\sigma}{\epsilon_0}$ where $\sigma = \frac{q}{area} = \int \frac{I}{area} dt = \frac{It}{\pi a^2}$

$$E = \frac{It}{\epsilon_0 \pi a^2}$$

(b) Energy density $u_{EM} = \frac{1}{2}\left(\epsilon_0 E^2 + B^2/\mu_0\right)$

$$u_{EM} = \frac{1}{2}\left(\epsilon_0 \frac{I^2 t^2}{\epsilon_0^2 \pi^2 a^4} + \frac{\mu_0^2 I^2 s^2}{4\pi^2 a^4 \mu_0}\right) = \frac{I^2 \mu_0}{2\pi^2 a^4}\left(\frac{t^2}{\epsilon_0 \mu_0} + \frac{s^2}{4}\right)$$

$$u_{EM} = \frac{I^2 \mu_0}{2\pi^2 a^4}\left[(ct)^2 + \left(\frac{s}{2}\right)^2\right], \quad \frac{\partial u_{EM}}{\partial t} = \frac{I^2 \mu_0 c^2 t}{\pi^2 a^4}$$

$$\vec{S} = \frac{E \times B}{\mu_0} = \frac{It}{\mu_0 \epsilon \pi a^2} \frac{\mu_0 I s}{2\pi a^2} = \frac{\mu_0 I^2 t s c^2}{2\pi^2 a^4}$$

$$\vec{\nabla} \cdot \vec{S} = \frac{\mu_0 I^2 t c^2}{2\pi^2 a^4} \frac{\partial s}{\partial s} = \frac{\mu_0 I^2 t c^2}{2\pi a^2} = \frac{\partial u_{EM}}{\partial t} \checkmark$$

$$\text{Power} = \int S \cdot da = \frac{\mu_0 I^2 t c^2}{2\pi^2 a^4} a \cdot 2\pi a w = \frac{\mu_0 I^2 t c^2 w}{\pi a^2} = \frac{I^2 t w}{\epsilon_0 \pi a^2}$$